A Lagrangian Relaxation Based Solution Approach for Finding the Most Reliable Path With and Without Link Travel Time Correlation

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Introduction

The shortest path problem has been extensively studied as a core subroutine for solving a wide variety of transportation network optimization problems, for example, traffic assignment and network design. Criteria to be considered in the shortest path problem include physical distance, travel time, as well as monetary cost along a path, and the corresponding generalized cost functions are typically additive. Many recent studies suggest that travel time reliability is an important element of a traveler's route and departure time scheduling. To assist public transportation planning agencies on evaluating strategies for improving network-wide traffic reliability, advanced traffic assignment tools are required to (1) produce reliability measures and (2) represent travel pattern shifts due to the change in system reliability. In addition, emerging personal navigation systems and pre-trip planning applications also start to deliver a richer set of route guidance information to assist commuters in finding reliable paths. In order to meet the above functional requirements, this study focuses on developing efficient problem reformulation and algorithms for solving the most reliable path problem.

A wide range of definitions and models have been proposed to measure and optimize travel time reliability. There are several types of travel time reliability measures commonly considered in transportation applications: (1) 90th- or 95th-percentile travel time, buffer and planning time index, (2) on-time arrival probability, (3) the travel time variation expressed in terms of standard deviation or coefficient of variation. In the first type of definitions, by generating the probability distribution of the travel time from the historical data, the 90th- or 95th-percentile travel time is defined as the travel time within which 90th- or 95th-percentile trips are completed. For travelers' convenience, the buffer and planning time indexes are calculated from the 90th- or 95th-percentile travel time to suggest the extra travel time that the traveler should buffer to ensure an on-time arrival. The second definition uses the percentage of trips that are completed within a reasonable buffered travel time (e.g. average travel time plus 20% buffer). The third type of definition is also widely used on traveler information provision systems with estimations of expected travel time and its variability. Although the travel time reliability measure in terms of standard deviation is relatively difficult to communicate directly with travelers, this type of model has been well calibrated in many previous empirical studies (e.g. Small [1], Noland et al. [2], and Noland and Polak [3]) and adopted in several traffic assignment models (e.g. Zhou et al., [4]). In addition, there are several variants of travel time reliability models being considered in the route finding problem. For example, the travel time reliability was modeled as the expected travel time and Time at Risk by Lu et al. [5], where the Time at Risk is a combination of expected travel time and its standard deviation. Sen et al. [6] proposed their reliability model as a linear combination of mean and variance of travel time, and solved as a set of parametric 0-1 quadratic integer programs.

This study adopts the reliability measure as the standard deviation of travel time The most reliable path problem under consideration is to find a path that optimizes two objectives: mean travel time and its standard deviation:

$$\min\left\{\text{mean}+\beta\sqrt{\text{var}}\right\}$$

The most reliable path problem shown above is quite different from regular multi-objective shortest path problems, and there are many fundamental issues to be addressed in order to fulfill the requirements for systematic modeling methodologies and efficient solution algorithms. First, incorporating the standard deviation of path travel time leads to a non-linear and non-additive disutility function, while the widely-used label correcting or label setting shortest path algorithms are only applicable for linear additive objection functions. Second, the standard deviation term is in fact a concave function of path travel time, which cannot be appropriately handled by common convex function-based linear approximation techniques.

Essentially, solving the proposed most reliable path problem needs to address several computational issues due to the nonlinear, nonadditive and concave objective function with mean travel time and standard deviation as two criteria. There are a number of previous studies addressing those aspects individually. Henig [7] presented efficient approximate methods on the shortest path problem with two criteria, which are assumed to be quasiconcave or quasiconvex utility functions. Scott and Bernstein [8] developed an iterative solution method for the shortest path problem where the value of time function is nonlinear and non-decreasing. In their study, the Lagrangian relaxation technique was used to approximate the non-linear objective function. Recently, Tsaggouris and Zaroliagis [9] combined the Lagrangian Relaxation and Hull approach to solve a non-additive bi-objective shortest path problem with a non-linear, convex and non-decreasing cost function. For the concave property of cost function, Larsson et al. [10] provided a Lagrangian dualization based heuristic algorithm for the concave minimum cost network flow problem.

The remainder of this paper is structured as follows. The next section provides the problem statement and proposed two different models for the most reliable path problem: with and without link correlation. Then these two models are reformulated and solved using the Lagrangian relaxation technique to handle the nonadditive and nonconvex objective functions, while the sub-gradient method is used for iteratively finding the optimal Lagrangian multipliers. Finally, this paper evaluates the proposed algorithms through numerical experiments on large-scale networks.

Problem Statement and Model Assumptions

Notations

For easy reference, we first list all the notations that are used in this paper:

N = set of nodes
A = set of links
p = a feasible path
m = the number of links in a path
c_p = the travel time of path p
$\bar{c}_p$ = the mean travel time of path p
i, j = subscript for nodes
l = subscript for the index of a link in a path, l= 0, …, m-1
a_l = a link in path p, with index number l
a_{ij} = a directed link from node i to j
c_l = the travel time of link a_l, considered as a random variable
c_{ij} = the travel time of link a_{ij}, considered as a random variable
$\bar{c}_l$ = the mean travel time of link a_l
$\bar{c}_{ij}$ = the mean travel time of link a_{ij}
$f(c_l)$ = the probability distribution function of c_l
$f(c_{ij})$ = the probability distribution function of c_{ij}
σ_{ij}^2 = the variance of link travel time c_{ij}

x_{ij} = a binary variable indicates the selection of link a_{ij}

X = the set of target binary variables $\{x_{ij} \mid ij \in A\}$

D = set of travel time measurement samples
n = number of samples in set *D*
d = subscript for samples
$c_{p,d}$ = travel time of path *p* in sample *d*

$c_{l,d}$ = travel time of link *l* in sample *d*

$c_{ij,d}$ = travel time of link (*i, j*) in sample *d*

β = reliability coefficient

k = superscript for iterations in sub-gradient method

Problem Statement

Let G (***N***, ***A***) represent a transportation network, where *N* is the set of nodes and *A* is the set of links. Each link can be denoted either a directed link a_{ij} from node *i* to *j*, or an indexed link a_l in a path *p* with *m* links. Accordingly, the travel time of each link is denoted as c_{ij} or c_l. Consider binary variable set $x_{ij} \in \{0,1\}$ that indicates the selection of the link a_{ij} for the least travel time path, a least travel time problem for a pre-specified OD pair (*o*, *d*) can be described as

$$z^* = \min \sum_{ij \in A} c_{ij} x_{ij} \tag{1}$$

Subject to the following flow balance constraints

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = \begin{cases} 1 & i = o \\ 0 & i \in N - \{o,d\} \\ -1 & i = d \end{cases} \tag{2}$$

The above integer linear program that can be solved using regular label correcting or label setting shortest path algorithms (Ahuja et al. [11]). The desired binary variable set $x_{ij} \in \{0,1\}$ indicates the selection of the link a_{ij} for the least travel time path.

In order to incorporate the travel time variability into the objective function, we consider travel time of each link (*i*, *j*) as a random variable with mean travel time $\overline{c_{ij}}$ and probability distribution function $f(c_{ij})$. Furthermore, the travel time probability distribution of each link can be estimated from the historical travel time data, and it typically depends on a number of factors such as link type, volume to capacity ratio and so on. It should be noted that, the actual link travel time is a time-varying variable further depending on time of day, day of week and month of year. In this study, we focus on the static shortest path problem and consider link travel times are static parameters. The time-varying least travel time problem has been addressed by many studies (e.g. Miller-Hooks and Mahmassani [12]).

This paper considers the following most reliable path problem as an integer non-linear minimization problem (P) by combining mean path travel time and its standard deviation:

$$z^* = \min \sum_{ij \in A} \bar{c}_{ij} x_{ij} + \beta \sqrt{\mathrm{var}(c_p)} \tag{3}$$

Subject to

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b \tag{4}$$

where $b = \begin{cases} 1 & i = o \\ 0 & i \in N - \{o, d\} \\ -1 & i = d \end{cases}$ represents the flow status for each node i in the network, and β is the reliability coefficient that reflects the significance of travel time variability. Commonly it can be derived as the ratio of Value of Reliability (VOR) to Value of Time (VOT), in which way normalized the reliability into time unit. The reliability coefficient could vary across different travelers and different trip purposes (e.g. business trip vs. recreational trip).

Given link travel time data, the calculation of mean path travel time for a path p is straightforward: $\bar{c}_p = \sum_{l=1}^{m} \bar{c}_l$, while the variance of path travel time can be expressed as

$$\begin{aligned} \mathrm{var}(c_p) &= \int_0^{+\infty} \left(c_p - \bar{c}_p\right)^2 f(c_p) dc_p \\ &= \int_{c_m=0}^{+\infty} \cdots \int_{c_2=0}^{+\infty} \int_{c_1=0}^{+\infty} \left(\sum_{l=1}^{m} c_l - \sum_{l=1}^{m} \bar{c}_l \right)^2 f(c_1, c_2, \ldots, c_m) dc_1 dc_2 \ldots dc_m \end{aligned} \tag{5}$$

Obviously, it is computationally intractable to obtain the multi-dimension probability distribution function $f(c_1, c_2, \ldots, c_m)$ for each path p. Thus, two different methods are considered below to calculate the path travel time variance with and without link travel time correlation assumptions.

Model 1: Independent Distribution Based Model

One simple approach on calculating Eq. (55) is to assume no spatial correlation for travel times on different links. That is, by assuming independent distributions among link travel times $c_1, c_2, \ldots, c_m$, the expression of variance in Eq. (55) can be reduced to

$$
\begin{aligned}
\operatorname{var}(c_p) &= \sum_{l=0}^{m-1} \int_{c_l=0}^{+\infty} \left(c_l - \overline{c}_l\right)^2 f(c_1, c_2, ..., c_m) dc_l \\
&= \sum_{l=1}^{m} \operatorname{var}(c_l)
\end{aligned}
\tag{6}
$$

In the above equation, the variance of path travel time is represented as a sum of independent link travel time variances, which are relatively easy to obtain from existing historical database. Denoting independent link variance as σ_{ij}^2 , the most reliable path problem with independent link travel time distribution is then formulated as

$$
\begin{aligned}
& z_1{*} = \min \sum_{ij \in A} c_{ij} x_{ij} + \beta \sqrt{\sum_{ij \in A} \sigma_{ij}^2 x_{ij}} \\
& \text{s.t.} \qquad \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b
\end{aligned}
\tag{7}
$$

Model 2: Sampling-based Model

In reality, travel times among different links could be highly correlated, e.g. due to the propagation of congestion from a downstream link to an upstream link along a freeway or arterial corridor. In order to explicitly allow the link correlation in path travel time variable calculation, a sampling-based approximation method is used in this study to more systematically formulate the most reliable path problem.

According to the Monte Carlo method, a continuous stochastic problem can be approximated as a discrete problem by taking *n* samples from the random variable:

$$
\begin{aligned}
\operatorname{var}(c_p) &\approx \frac{1}{n-1} \sum_{d=1}^{n} \left(c_{p,d} - \overline{c}_p\right)^2 \\
&= \frac{1}{n-1} \sum_{d=1}^{n} \left(\sum_{l=1}^{m} c_{l,d} - \sum_{l=1}^{m} \overline{c}_l \right)^2
\end{aligned}
\tag{8}
$$

That is to say, one can take *n* days' samples from a multi-day historical database and use them directly to calculate the variance of path travel time. By doing so, the correlation among link travel times has been automatically represented by the sample set, without explicitly requiring the variance-covariance matrix.

According to the sampling approach in Eq. (88), the most reliable path problem with link travel time correlation is formulated as

$$z_2{*} = \min \sum_{ij \in A} \overline{c}_{ij} x_{ij} + \beta \sqrt{\frac{1}{n-1} \sum_{d=1}^{n} \left(\sum_{ij \in A} c_{ij,d} x_{ij} - \sum_{ij \in A} \overline{c}_{ij} x_{ij} \right)^2} \tag{9}$$

$$\text{s.t.} \quad \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b$$

It should be remarked that, the mean travel time for each link in this model can be also derived from different samples, i.e.: $\overline{c}_{ij} = \frac{1}{n} \sum_{d=1}^{n} c_{ij}^{d}$.

Compared to the first model, this sampling-based model needs larger sample size and longer computational time to achieve acceptable level of accuracy. For applications lacking sufficient link travel time measurements, the first model is still a feasible and compromising option, although it might not find the most reliable path that recognizes link travel time correlation.

Solution Approaches

In this section we will discuss a Lagrangian Relaxation based solution methodology for finding the most reliable path with and without link travel time correlation assumptions.

Approach for Independent Distribution Based Model

In the proposed model with the independent distribution assumption, the total travel time variance of a path is calculated by summing up the variances of each individual link along the path, as shown in the optimization program(77). Since the standard deviation component is non-additive along different links of a path, we apply a Lagrangian relaxation-based transformation to construct an approximate linear program in order to apply the regular shortest path algorithm for additive linear objective functions.

Lagrangian Relaxation

To remove the non-additivity on target binary variable *x*, we first introduce a positive auxiliary variable *y* to the program (77) to move the variance term to a constraint:

$$\min \sum_{ij \in A} c_{ij} x_{ij} + \beta \sqrt{y} \tag{10}$$

s.t.

$$\sum_{ij \in A} \sigma_{ij}^2 x_{ij} = y \tag{11}$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b$$

Since $U(y) = \sqrt{y}$ is a concave, monotonically increasing function, in order to use the standard Lagrangian Relaxation described in [9], we further relax the equality constraint (1111) and substitute it by an inequality:

$$\sum_{ij \in A} \sigma_{ij}^2 x_{ij} \le y \tag{12}$$

Now the optimization program (77) becomes a constrained shortest path problem with a linear cost function and a linear constraint for path variance. To further remove constraint (1212), we introduce a Lagrangian multiplier $\mu \ge 0$ and bring the explicit linear constraint back to the original objective function (1010):

$$\min \sum_{ij \in A} c_{ij} x_{ij} + \beta\sqrt{y} + \mu\left(\sum_{ij \in A} \sigma_{ij}^2 x_{ij} - y\right) \tag{13}$$

s.t. $\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b$

The above minimization problem is linear in terms of the targeted binary variable set **X**, which is referred as a Lagrangian relaxation or sub-problem of the original problem (77). In a Lagrangian relaxation solution framework, the original problem P is normally referred as the primal problem and the Lagrangian transformation as the dual problem. By re-grouping variables, a Lagrangian function is constructed as:

$$L(\mu) = \min\left\{\sum_{ij \in A}\left(c_{ij} + \mu\sigma_{ij}^2\right)x_{ij} + \beta\sqrt{y} - \mu y : \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b\right\} \tag{14}$$

In the above function, the primal variable x has a linear cost function combined from link travel time and weighted link variance. In other words, we approximate the travel time and standard deviation combined routing problem (77) (primal problem) as a linear cost function routing problem (dual problem), which can be easily solved using label-correcting or label-setting shortest path algorithms. For best approximation, the Lagrangian multiplier μ and the auxiliary variable y need to be determined wisely so that the gap between the objective value in the dual and primal problems can be minimized.

In equation(1313), to relax the inequality constraint(1212), a non-positive portion was added to the objective function. This relaxation shows that for each positive value of the Lagrangian multiplier, the corresponding value of the Lagrangian function $L(\mu)$ is providing a lower bound for the optimal objective function value z^* of the primal problem [9]. Therefore, to approach the objective value in the primal problem, the best lower bound needs to be determined. Let's denote L^* to be the maximum value of $L(\mu)$ according to μ, i.e.:

$$L^* = \max_{\mu} L(\mu) \tag{15}$$

The difference between the primal optimal value z^* and the dual optimal value L^* is called the duality gap. By achieving L^*, the primal variable **X** calculated with corresponding μ in the dual problem will provide an approximate solution for the primal problem(77).

Dual Function Decomposition

Since the only constraint in the dual problem is the flow balance constraint about primal variable **X**, for any particular Lagrangian multiplier, the Lagrangian function (1414) can be decomposed into and solved by two independent sub-functions:

$$L(\mu) = L_x(\mu) + L_y(\mu) \tag{16}$$

The first sub-function $L_x(\mu)$ only contains the variable set **X**:

$$L_x(\mu) = \min\left\{\sum_{ij\in A}\left(c_{ij} + \mu\sigma_{ij}^2\right)x_{ij} : \sum_{j:ij\in A} x_{ij} - \sum_{j:ji\in A} x_{ji} = b\right\} \tag{17}$$

This is a linear function for x and thus can be solved by using the shortest path algorithm with combined link costs $c_{ij} + \mu\sigma_{ij}^2$.

The second sub-function $L_y(\mu)$ is a minimization problem with respect to auxiliary variable y. By calculating the second order derivative of $L_y(\mu)$, it is easy to verify that the second part of the dual function

$$L_y(\mu) = \min\left\{\beta\sqrt{y} - \mu y\right\} \tag{18}$$

is a concave function in terms of y. To solve the above single-variable concave minimization problem, the feasible region of y needs to be determined, because the optimal value is attained at one of the extreme points of the feasible region (e.g. Larsson [10]).

The auxiliary variable y corresponds to the variance of the path travel time. As illustrated in Figure 1, in order to minimize the total path cost, the travel time variance for the most reliable path should be no larger than the variance for the least travel time path. Therefore, the feasible region of y is can be determined as an interval between zero and the variance of the least travel time path.

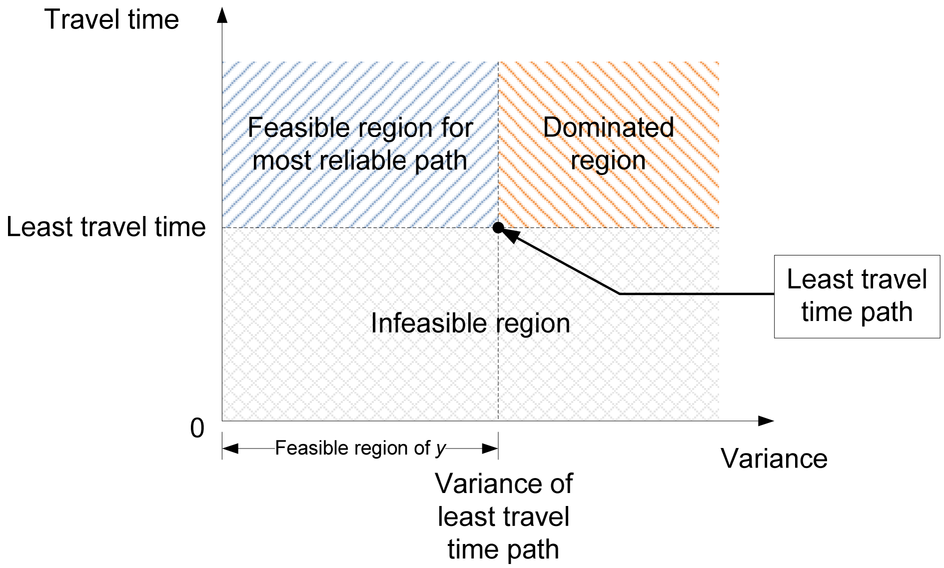

Figure 1: Feasible region of *y*

Sub-gradient Method

To find the optimal Lagrangian multiplier μ that maximizing $L(\mu)$, an iterative sub-gradient method is used in this study. Suppose the Lagrangian function is differentiable, the search direction of μ in the optimization process can be determined from the gradient of $L(\mu)$ with respect to μ:

$$\nabla L(\mu) = \sum_{ij \in A} \sigma_{ij}^2 x_{ij} - y \tag{19}$$

Starting from any feasible initial choice of the Lagrangian multiplier μ_0, for any μ^k at iteration k, the sub-functions (1717) and (1818) are solved and the solutions are denoted as x_{ij}^k and y^k, respectively. Then we iteratively calculate the updated value of Lagrangian multiplier as follows:

$$\mu^{k+1} = \mu^k + \theta^k \left(\sum_{ij \in A} \sigma_{ij}^2 x_{ij}^k - y^k\right) \tag{20}$$

Until the step size parameter θ^k is smaller than a marginal bound or the iteration k is larger than a pre-defined value.

As discussed in [9], the selection of step size θ^k needs to satisfy:

$$\theta^k \to 0 \quad \text{and} \quad \sum_{j=1}^{k} \theta^j \to \infty \tag{21}$$

In practice, heuristic algorithms are normally used. One popular implemented algorithm is:

$$\theta^k = \frac{\lambda^k \left[UB - L(\mu^k) \right]}{\left\| \sum_{ij \in A} \sigma_{ij}^2 x_{ij}^k - y^k \right\|^2} \tag{22}$$

In this expression, UB is the upper bound of the optimal objective function value z^* in the primal problem, and λ^k is a scalar chosen between 0 and 2, which is for the step size optimization problem only.

Algorithm

The algorithm for solving a most reliable path problem without link travel time correlation is illustrated in Figure 2:

Step 1: Initialization

Choose an initial Lagrangian multiplier $\mu > 0$

Find the travel time and variance for the least travel time path

Step 2: Solve dual problem

Solve $L_x(\mu)$ with the shortest path algorithm

Solve $L_y(\mu)$ by using two extreme points: 0 and variance of the least travel time path

Step 3: Update Lagrangian multiplier

Calculate Lagrangian multiplier with Eq. (2020) and Eq. (2222)

Step 4: Termination rule test

If $\theta^k < \varepsilon$ or $k > K_{\max}$, terminate program. (ε is minimum step size, $K_{\max}$ is maximum iteration)

Else, go to Step 2.

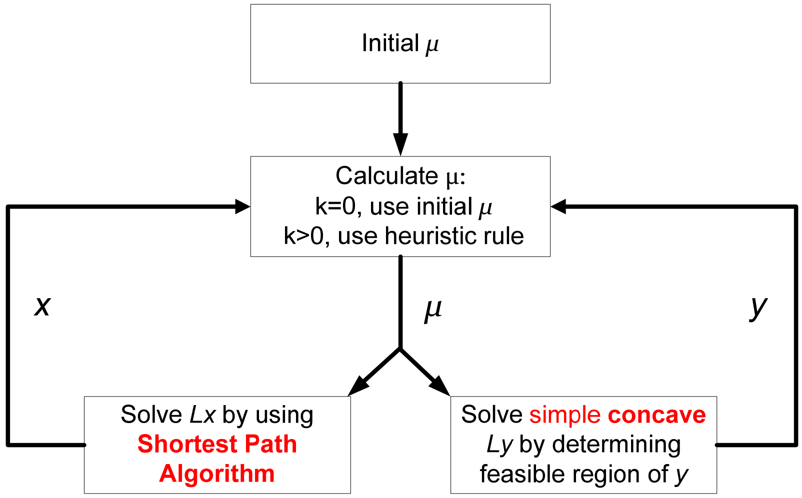

Figure 2: Algorithm for independent distribution based model

Approach for Sampling-based Model

In last section we solved the most reliable path search model without travel time correlation. In order to take into account of the link travel time correlation, a Monte Carlo based approximation method is used to propose the sampling-based model in Eq. (99). In this model, given the same transportation network G (***N***, ***A***), we generate a sample set ***D*** with n travel time measurements from the same time at the same day-of-week. The sample domain is denoted with the subscript d for variables. Similar to the independent distribution based model, because of the non-linear and non-additivity of the objective function, a Lagrangian relaxation is applied to solve the sampling-based model.

Lagrangian Relaxation

In order to approximate the minimization problem(99) with a linear optimization problem, we implement a two-step Lagrangian relaxation with two sets of auxiliary variables:

$$\sum_{ij\in A} c_{ij,d} x_{ij} - \sum_{ij\in A} \bar{c}_{ij} x_{ij} = w_d \qquad \forall d \in D \tag{23}$$

$$\frac{1}{n-1}\sum_{d=1}^{n} w_d^2 = y \tag{24}$$

After relaxing the equality constraints in (2323) and (2424) with inequality ones, the minimization problem can be reformulated as

$$z^* = \min \sum_{ij\in A} \bar{c}_{ij} x_{ij} + \beta\sqrt{y} \tag{25}$$

s.t.

$$\sum_{ij \in A} c_{ij,d} x_{ij} - \sum_{ij \in A} \overline{c}_{ij} x_{ij} \le w_d \qquad \forall d \in D \tag{26}$$

$$\frac{1}{n-1} \sum_{d=1}^{n} w_d^2 \le y \tag{27}$$

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b$$

In this reformulation *n*+1auxiliary variables are introduced. The variable w_d for each sample *d* represents the difference between the mean path travel time and the path travel time on sample *d*. While the variable *y* is the average path travel time deviation between samples and the sample mean, or the path travel time variance in other words.

To further remove constraints (2626) and (2727), a set of Lagrangian multipliers, denoted as μ_d and ν sequentially, is introduced in order to move the explicit inequality constraints into the objective function (2525):

$$\begin{aligned} & L(\mu_1, \ldots, \mu_n, \nu) \\ & = \min \left\{ \sum_{ij \in A} \overline{c}_{ij} x_{ij} + \beta \sqrt{y} + \sum_{d=1}^{n} \mu_d \left(\sum_{ij \in A} c_{ij,d} x_{ij} - \sum_{ij \in A} \overline{c}_{ij} x_{ij} - w_d \right) + \nu \left(\frac{1}{n-1} \sum_{d=1}^{n} w_d^2 - y \right) \right. \\ & \qquad \left. : \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b \right\} \end{aligned} \tag{28}$$

By regrouping the variables, we will have a clearer view on the components of the dual problem:

$$\begin{aligned} & L(\mu_1, \ldots, \mu_n, \nu) \\ & = \min \left\{ \sum_{ij \in A} \left[\left(1 - \sum_{d=1}^{n} \mu_d \right) \overline{c}_{ij} + \sum_{d=1}^{n} \mu_d c_{ij,d} \right] x_{ij} + \sum_{d=1}^{n} \left(\frac{1}{n-1} \nu w_d^2 - \mu_d w_d \right) + \beta \sqrt{y} - \nu y \right. \\ & \qquad \left. : \sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b \right\} \end{aligned} \tag{29}$$

The dual function (2929) has a linear objective function corresponding to the primal variable **X**. For each link, the cost function is a combination of weighted sample travel times on different days. By adjusting the Lagrangian multipliers μ_d and ν, we may achieve an optimal linear cost function that will maximize the dual function so as to best approximate the non-linear objective function in the primal problem.

Dual Function Decomposition

We decompose the dual function into a set of sub-functions:

$$L_x(\mu_1,...,\mu_n,\nu)=\min\left\{\sum_{ij\in A}\left[\left(1-\sum_{d=1}^{n}\mu_d\right)\overline{c}_{ij}+\sum_{d=1}^{n}\mu_d c_{ij,d}\right]x_{ij}:\sum_{j:ij\in A}x_{ij}-\sum_{j:ji\in A}x_{ji}=b\right\} \quad (30)$$

$$L_{w_d}(\mu_1,...,\mu_n,\nu)=\min\left\{\frac{1}{n-1}\nu w_d^2-\mu_d w_d\right\} \quad \forall d\in D \quad (31)$$

$$L_y(\mu_1,...,\mu_n,\nu)=\min\left\{\beta\sqrt{y}-\nu y\right\} \quad (32)$$

The first sub-function(3030) can be easily solved using shortest path algorithms. The second sub-function set(3131) contains one convex minimization problem for each auxiliary variable w_d and can be solved using first-order gradient, i.e.:

$$\begin{gathered}\frac{\partial L_{w_d}(\mu_1,...,\mu_n,\nu)}{\partial w_d}=\frac{2}{n-1}\nu w_d-\mu_d=0\\ w_d=\frac{\mu_d(n-1)}{2\nu}\end{gathered} \quad (33)$$

The third sub-function(3232) is a concave minimization problem for variable y. Since y represents the variance of the path travel time, the feasible region is between zero and the variance of the path with least travel time, and the minimization point locates at one of the extreme points of the feasible region.

Sub-gradient Method

The subgradient method is used here as well. The search direction for each Lagrangian multiplier is found using following equations.

$$\nabla L(\mu_1,...,\mu_n,\nu)=\left(\sum_{ij\in A}\left(c_{ij,1}-\overline{c}_{ij}\right)x_{ij}-w_1,...,\sum_{ij\in A}\left(c_{ij,n}-\overline{c}_{ij}\right)x_{ij}-w_n,\frac{1}{n-1}\sum_{d=1}^{n}w_d^2-y\right)$$

$$\mu_d^{k+1}=\mu_d^k+\theta_{\mu_d}^k\left(\sum_{ij\in A}\left(c_{ij,d}-\overline{c}_{ij}\right)x_{ij}^k-w_d^k\right) \quad \forall d\in D$$

$$\nu^{k+1}=\nu^k+\theta_\nu^k\left(\frac{1}{n-1}\sum_{d=1}^{n}\left(w_d^k\right)^2-y^k\right)$$

The step size of each iteration k is calculated using heuristic algorithms:

$$\theta_{\mu_d}^k = \frac{\lambda_{\mu_d}^k \left[L_{UB}(\mu_1,...,\mu_n,\nu) - L(\mu_1^k,...,\mu_n^k,\nu^k) \right]}{\left\| \sum_{ij \in A} \left(c_{ij,d} - \overline{c}_{ij} \right) x_{ij}^k - w_d^k \right\|^2} \qquad \forall d \in D$$

$$\theta_\nu^k = \frac{\lambda_\nu^k \left[L_{UB}(\mu_1,...,\mu_n,\nu) - L(\mu_1^k,...,\mu_n^k,\nu^k) \right]}{\left\| \frac{1}{n-1} \sum_{d=1}^{n} \left(w_d^k \right)^2 - y^k \right\|^2}$$

Numerical Experiments

Two illustrative examples are shown below for the proposed two most reliable path models, respectively.

Independent Distribution Based Model

Suppose an origin-destination pair with three paths available is considered with following path travel time data:

Table 1: Path travel time data for independent distribution based model

Path	Mean Travel Time	Travel Time Variance	Travel Time Standard Deviation	Value of Objective Function (β=1)
A	35	0	0	35
B	29	49	7	36
C(opt)	31	4	2	33

By simply calculating the values of objective function, it is obvious that path C is the most reliable path with the independent distribution based model. Figure 3 shows the relationship between the Lagrangian multiplier and the value of objective function.

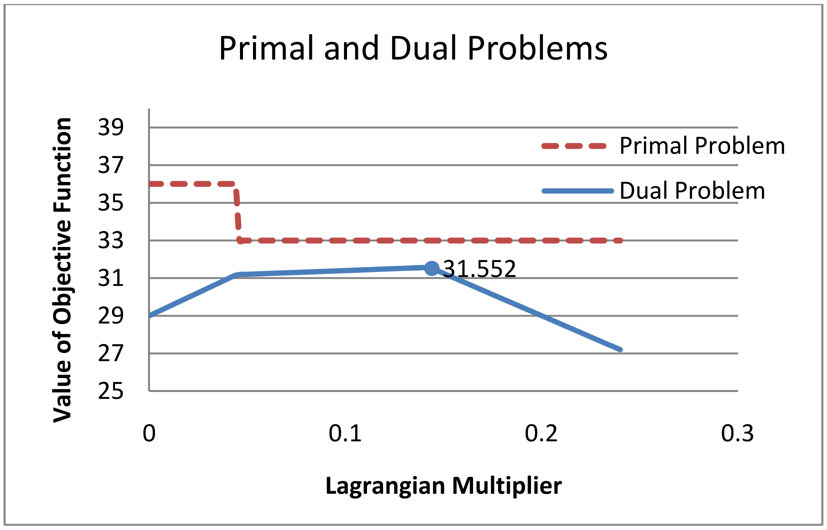

Figure 3: Solution results for independent distribution based model

When the Lagrangian multiplier equals to 31.552, path C is the optimal solution for the dual problem, which is also the most reliable path as in the primal problem. Note that, there still exists a duality gap between primal and dual problems, due to the approximation nature of the Lagrangian relaxation method.

Sampling-based Model

In this numerical example, an origin-destination pair with two possible paths is considered. Each path is constructed with two links. Four-day samples are given in Table 2.

Table 2: Sample data for sampling-based model

Link	Day 1	Day 2	Day 3	Day 4	Mean Travel Time	Travel Time Variance
A1	2	1	2	2	1.75	0.25
A2	1	2	1	2	1.5	0.33
B1	2	2	1	1	1.5	0.33
B2	2	2	1	1	1.5	0.33

It is easy to verify that path A is the most reliable path (Table 3):

Table 3: Path travel time data for sampling-based model

Path	Day 1	Day 2	Day 3	Day 4	Mean Travel Time	Travel Time Variance	Value of Objective Function (β=1)
A (opt)	3	3	3	4	3.25	0.25	3.75
B	4	4	2	2	3	1.33	4.15

The sampling-based model can be solved with the optimal value for the dual problem is 3.15. Under the optimal value set of Lagrangian multipliers, the dual problem was solved with solution of path A. In Figure 4, we plotted the change of dual function value according to Lagrangian multipliers (when $\mu_1 = \mu_2, \mu_3 = \mu_4$)

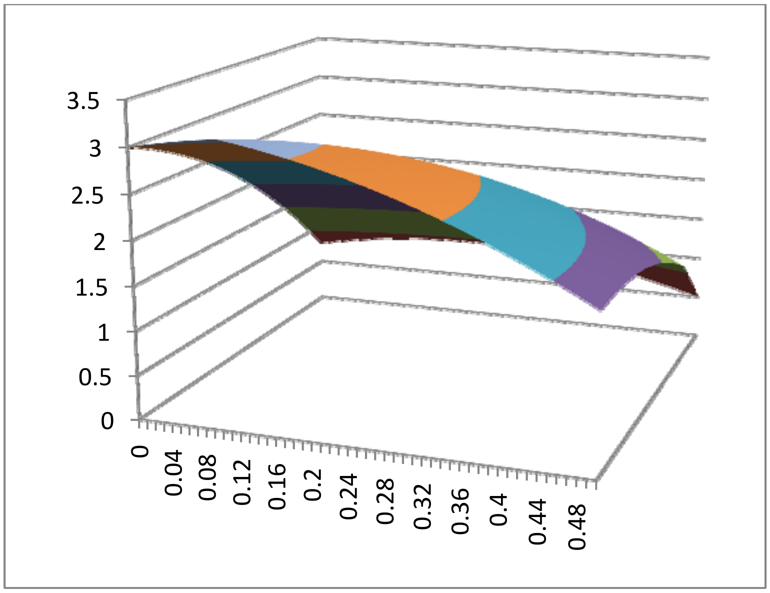

Figure 4: Change of dual function value according to Lagrangian multipliers

Conclusions and Further Study

This study proposed two models for the most reliable path problem with and without link travel time correlation. Then Lagrangian Relaxation based solution approaches and algorithms are provided for each model to solve the most reliable path problem under different assumptions. The illustrative examples show that by solving the dual problems, we can approximate the optimal solution for the primal problems.

Reference

1. Small K. A. (1982). The Scheduling of Consumer Activities: Work Trips. *American Economic Review* Vol. 72(3), pp. 467-479.
2. Noland, R. B., K. A. Small, P. M. Koskenoja, and X. Chu. (1998). Simulating Travel Reliability. *Regional Science and Urban Economics*, Elsevier, Vol. 28(5), pp. 535-564.
3. Noland, R.B., and Polak, J. W. (2002). Travel time variability: a review of theoretical and empirical issues. *Transportation Reviews* 122 (1), pp. 39-54.

4. Zhou, X, Mahmassani, H.S. and Zhang, K. (2008) Dynamic Micro-assignment Modeling Approach for Integrated Multimodal Urban Corridor Management. Accepted for publication in *Transportation Research Part C*. Vol. 16, No. 2, pp. 167-186.
5. Lu, J.G., Ban, X., Qiu, Z.J., Yang, F., and Ran, B. (2005) Robust Route Guidance Model Based on Advanced Traveler Information Systems. *Transportation Research Record* 1935, 1-7.
6. Suvrajeet Sen, Rekha Pillai, Shirish Joshi, Ajay K. Rathi. (2001). A Mean-Variance Model for Route Guidance in Advanced Traveler Information Systems, *Transportation Science*, Vol.35 n.1, pp.37-49, February.
7. Henig, Mordechai I., (1986). The shortest path problem with two objective functions, *European Journal of Operational Research*, Elsevier, Vol. 25(2), pages 281-291, May.
8. K. Scott and D. Bernstein, (1997). Solving a Best Path Problem when the Value of Time Function is Nonlinear, preprint 980976 of the *Transportation Research Board*.
9. G. Tsaggouris and C. Zaroliagis, (2004). Non-Additive Shortest Paths, *Algorithms - ESA 2004*, LNCS Vol. 3221 (Springer-Verlag), pp. 822-834.
10. Larsson, Torbjorn & Migdalas, Athanasios & Ronnqvist, Mikael. (1994). A Lagrangean heuristic for the capacitated concave minimum cost network flow problem, *European Journal of Operational Research*, Elsevier, vol. 78(1), pages 116-129, October.
11. Ahuja, R.K., Magnanti, T.L., and Orlin J.B., (1993). *Network Flow: Theory, Algorithms and Applications*, Prentice Hall, N.J.
12. Miller-Hooks, E. and H. Mahmassani, (2000). Least Expected Time Paths in Stochastic, Time-Varying Transportation Networks, *Transportation Science* Vol.34, pp. 198-215