Efficient specification and Estimation of Choice Models in Activity-Travel Model Systems

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Presentation outline

- ✓ Introduction
- ✓ Discrete-Continuous
- ✓ Spatial Dependency
- ✓ Emerging Estimation Technique for discrete choice models
- ✓ Conclusion

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Introduction - Discrete-Continuous frameworks (D/C)

- Characterized by a continuous variable related to a discrete variable
  - Selective sample observation effect: Continuous outcome observed only if a discrete condition is met.

    Examples: Household income observation, GPS-based data
  - Endogenous treatment effect: The continuous equation depends on a discrete explanatory variable that is determined endogenously with the continuous variable. Examples: job training program – wages, seat belt use – injury severity

Introduction – Spatially and Socially dependent choice processes (SD)

- Characterized by a choice process influenced by unobserved error dependency based on spatial location
  - Spatial dependence across alternatives
  - Spatial dependence across observational units

- Tendency of data points to be similar when closer in space
  - Diffusion effects
  - social interaction effects
  - unobserved location-related effects

  - Examples: Residential location choice, Physical activity participation

State of the field – D/C

- Discrete choice models have seen substantial advancement in recent years
  - Mixed logit and advances in simulation
- Not the same level of maturity in discrete-continuous frameworks
  - "The field is still expanding more than it is coalescing" - Train
- Approaches
  - Heckman or Lee's approach
  - Semi-parametric and Non-parametric approaches
  - More recently Copula approach

State of the field – SD

- Spatial correlation across alternatives: choices correspond to spatial units.
  - Transportation and geography literature.
  - Common model structures include mixed logit, multinomial probit, GEV-based spatially correlated models.

- Spatial correlation across observational units: choices among the aspatial alternatives may be moderated by space.
  - Regional science and political science literature
  - Common model structure include Binary spatial probit model estimated using McMillen's EM, LeSage's MCMC etc.

D/C frameworks

- Direct and indirect utility approaches to modeling discrete/continuous frameworks
  - Typically D/C approaches begin with constrained direct utility functions
  - This constrained direct utility function can be equivalently represented by an indirect utility function
  - Once an indirect utility function is chosen, deriving demand functions is relatively easy
  - However, recently studies have started employing direct utility functions to model D/C frameworks particularly for multiple-discrete choices
  - An explicit framework employing direct utility functions applicable to multiple discrete problems is discussed in detail

Why multiple-discreteness

- Several consumer demand choices are characterized by multiple discreteness
  - Vehicle type holdings and usage
  - Household consumption patterns on consumer services/goods
  - Activity type choice and duration of participation
  - Airline fleet mix and usage
  - Carrier choice and transaction level
  - Brand choice and purchase quantity
  - Stock choice and investment amount

- Modeling methodologies of multiple discrete situations

  - Traditional random utility-based (RUM) single discrete choice models
    - Number of composite alternatives explodes with the number of elemental alternatives

  - Multivariate probit (logit) methods
    - Not based on a rigorous underlying utility-maximizing framework of multiple discreteness

  - Other issues with these methods
    - Cannot accommodate the diminishing marginal returns (*i.e.*, satiation) in the consumption of an alternative
    - Cumbersome to include a continuous dimension of choice

- Modeling methodologies of multiple discrete situations

  - Two alternative methods proposed by Wales and Woodland (1983)
    - Amemiya-Tobin approach
    - Kuhn-Tucker approach

  - Both approaches assume a direct utility function *U(x)* that is assumed to be quasi-concave, increasing, and continuously differentiable with respect to the consumption quantity vector *x*

  - Approaches differ in how stochasticity, non-negativity of consumption, and corner solutions (*i.e.*, zero consumption of some goods) are accommodated

- Methods proposed by Wales and Woodland
  - Amemiya-Tobin approach
    - Extension of the classic microeconomic approach of adding normally distributed stochastic terms to the budget-constrained utility-maximizing share equations
    - Direct utility function *U(x)* assumed to be deterministic by the analyst, and stochasticity is introduced post-utility maximization
  - Kuhn-Tucker (KT) approach
    - Based on the Kuhn Tucker or KT (1951) first-order conditions for constrained random utility maximization
    - Employs a direct stochastic specification by assuming the utility function *U(x)* to be random (from the analyst's perspective) over the population
    - Derives the consumption vector for the random utility specification subject to the linear budget constraint by using the KT conditions for constrained optimization
    - Stochastic nature of the consumption vector in the KT approach is based fundamentally on the stochastic nature of the utility function

- Advantages of KT approach
  - Constitutes a more theoretically unified and consistent framework for dealing with multiple discreteness consumption patterns
  - Satisfies all the restrictions of utility theory
  - Stochastic KT first-order conditions provide the basis for deriving the probabilities for each possible combination of corner solutions (zero consumption) for some goods and interior solutions (strictly positive consumption) for other goods
  - Accommodates for the singularity imposed by the "adding-up" constraint
- Problems with KT approach used by Wade and Woodland
  - Random utility distribution assumptions lead to a complicated likelihood function that entails multi-dimensional integration

- Studies that used the KT approach for multiple discreteness
  - Kim *et al.* (2002)
    - Used the GHK simulator to evaluate the multivariate normal integral appearing in the likelihood function in the KT approach
    - Used a generalized variant of the well-known translated constant elasticity of substitution (CES) direct utility function
    - Not realistic for practical applications and is unnecessarily complicated
  - Bhat (2005)
    - Introduced a simple and parsimonious econometric approach to handle multiple discreteness
    - Based on the generalized variant of the translated CES utility function but with a multiplicative log-extreme value error term
    - Labeled as the multiple discrete-continuous extreme value (MDCEV) model
    - MDCEV model represents the multinomial logit (MNL) form-equivalent for multiple discrete-continuous choice analysis and collapses exactly to the MNL in the case that each (and every) decision-maker chooses only one alternative
  - Several studies in the environmental economics field
    - Phaneuf *et al.*, 2000; von Haefen *et al.*, 2004; von Haefen, 2003a; von Haefen, 2004; von Haefen and Phaneuf, 2005; Phaneuf and Smith, 2005
    - Used variants of the linear expenditure system (LES) and the translated CES for the utility functions, and used multiplicative log-extreme value errors

MDCEV
Functional form of utility function

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k} \psi_k \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\}$$

- $U(\boldsymbol{x})$ is a quasi-concave, increasing, and continuously differentiable function with respect to the consumption quantity vector *x*
- ψ_k , γ_k and α_k are parameters associated with good *k*

Assumptions

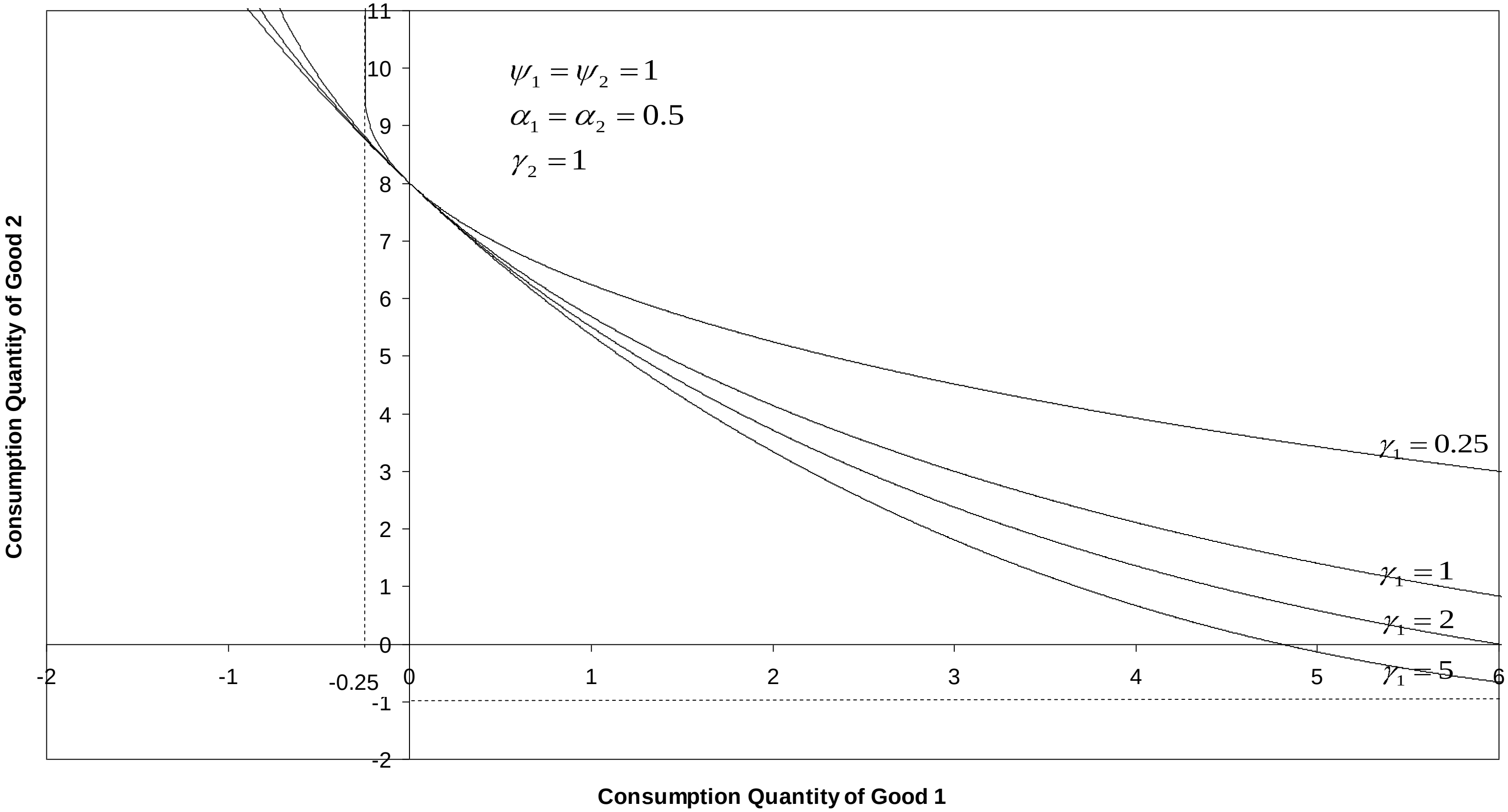
- Additive separability
  - All the goods are strictly Hicksian substitutes
  - Marginal utility with respect to any good is independent of the level of consumption of other goods
- Weak complementarity

Role of ψ_k

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k} = \psi_k \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k - 1}$$

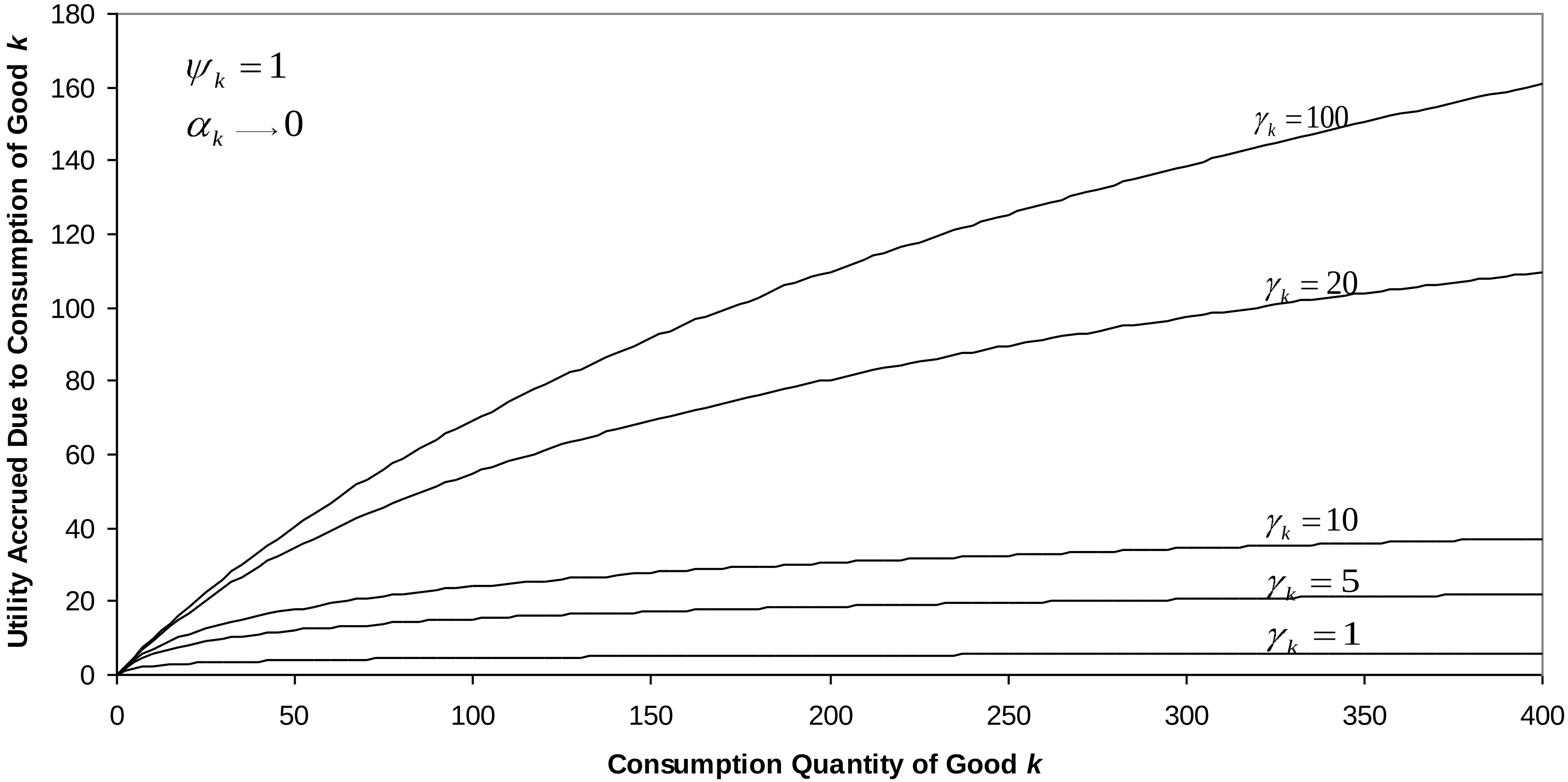
- ψ_k represents the baseline marginal utility, or the marginal utility at the point of zero consumption
- Higher baseline ψ_k implies less likelihood of a corner solution for good *k*

Role of γ_k



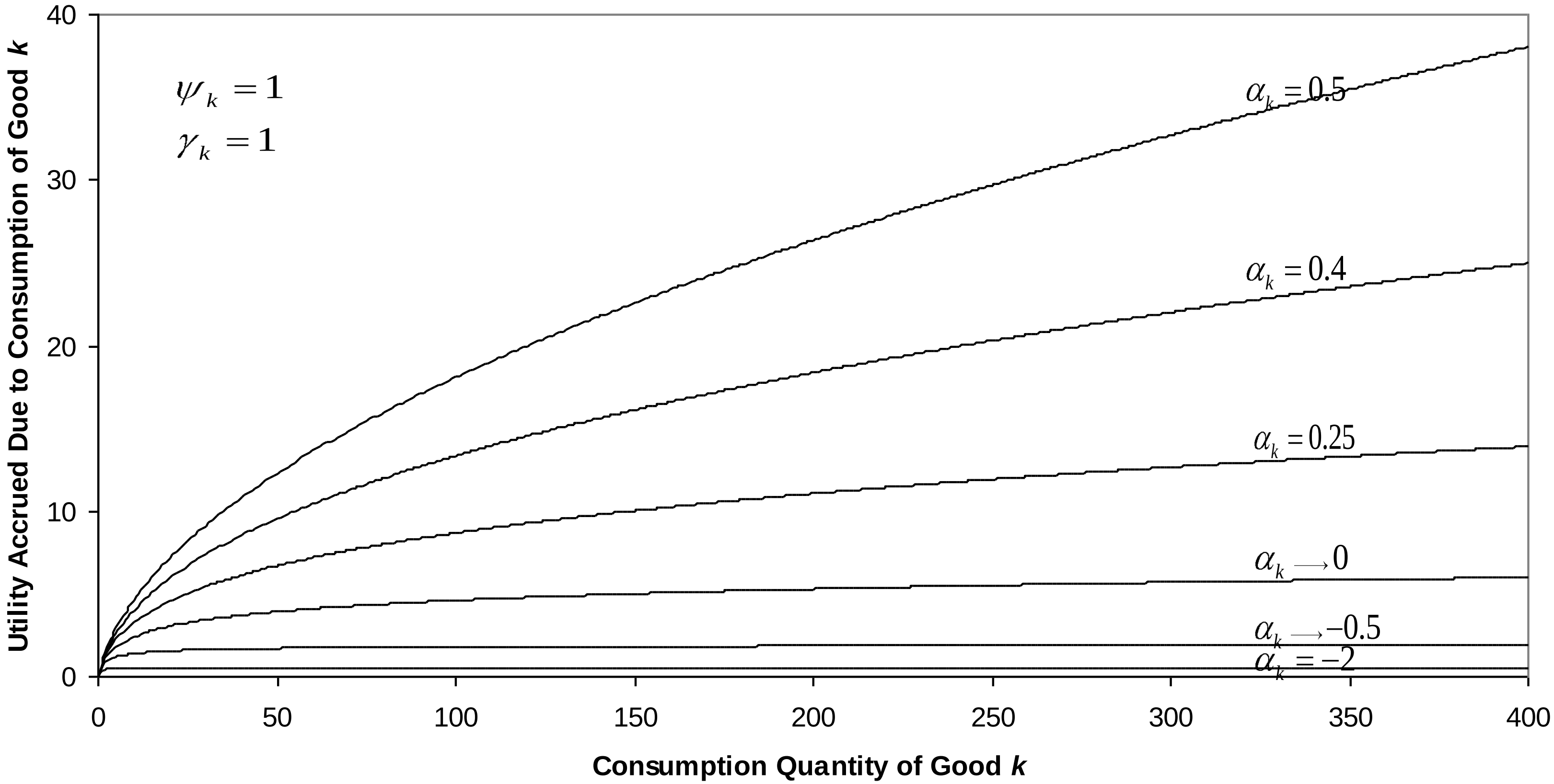
Indifference Curves Corresponding to Different Values of γ_1

Role of γ_k



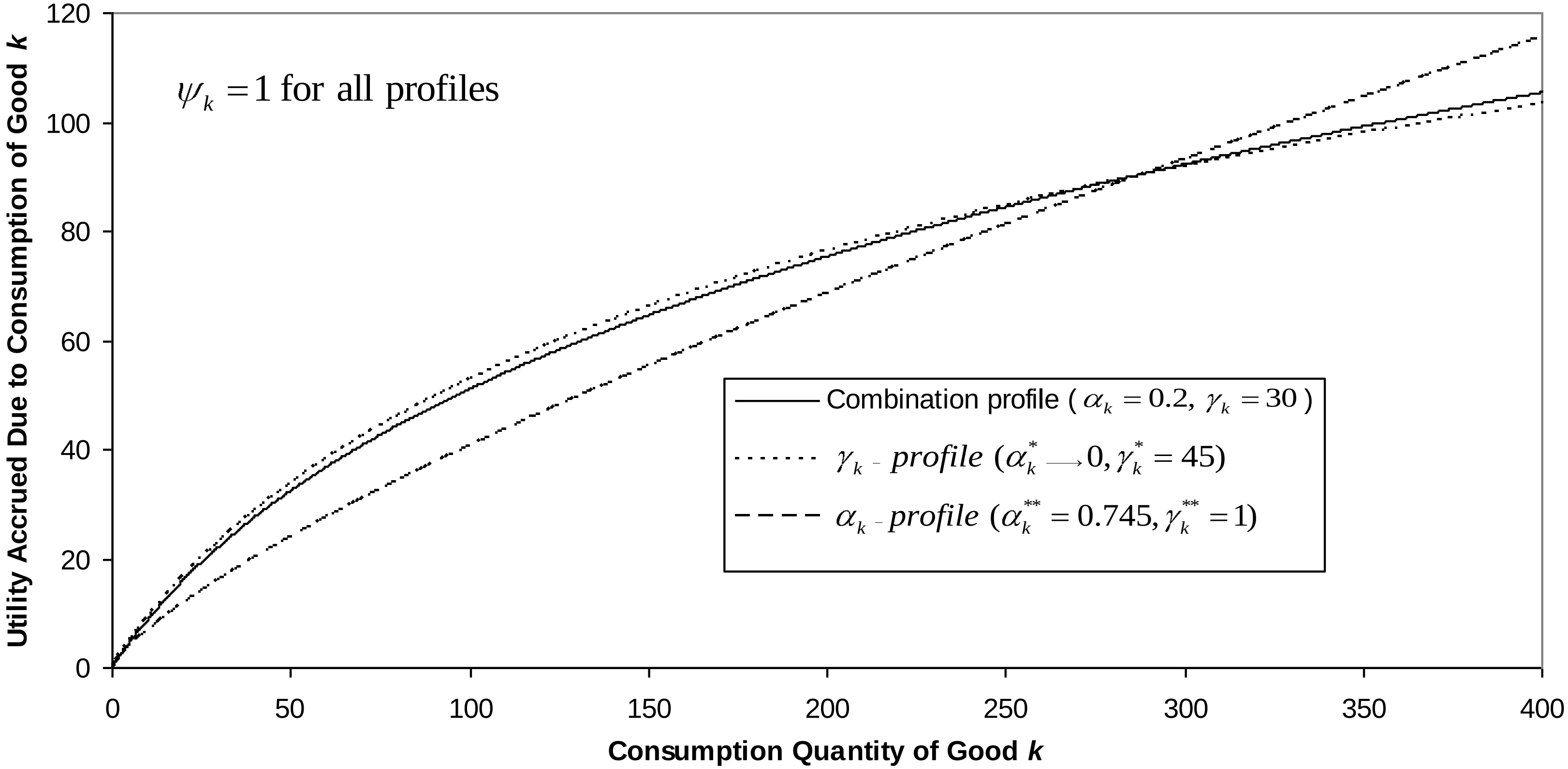
Effect of γ_k Value on Good *k*'s Subutility Function Profile

Role of α_k



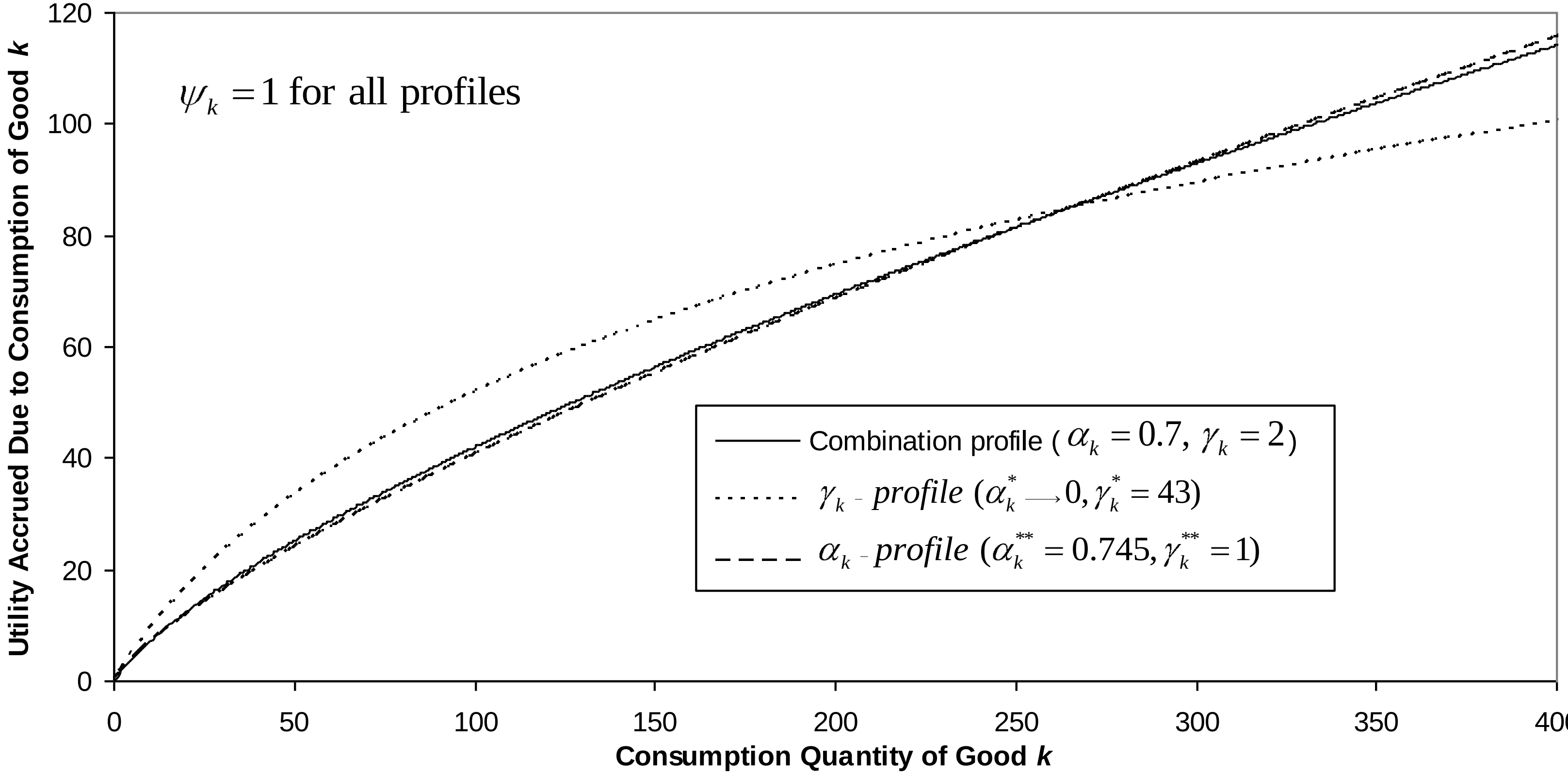
Effect of α_k Value on Good *k*'s Subutility Function Profile

Empirical identification issues associated with utility form



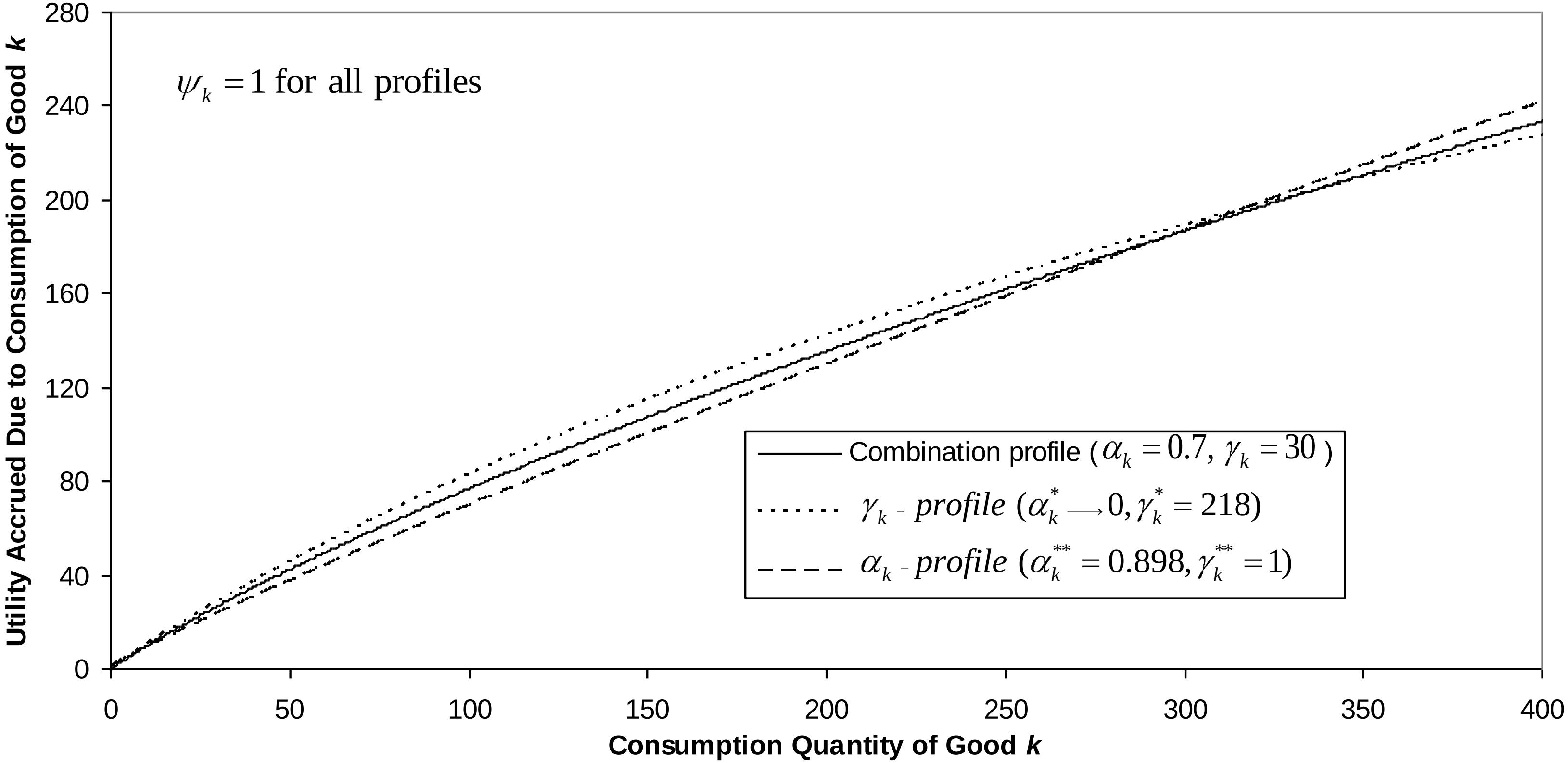
Alternative Profiles for Moderate Satiation Effects with Low α_k Value and High Value

Empirical identification issues associated with utility form-cont'd



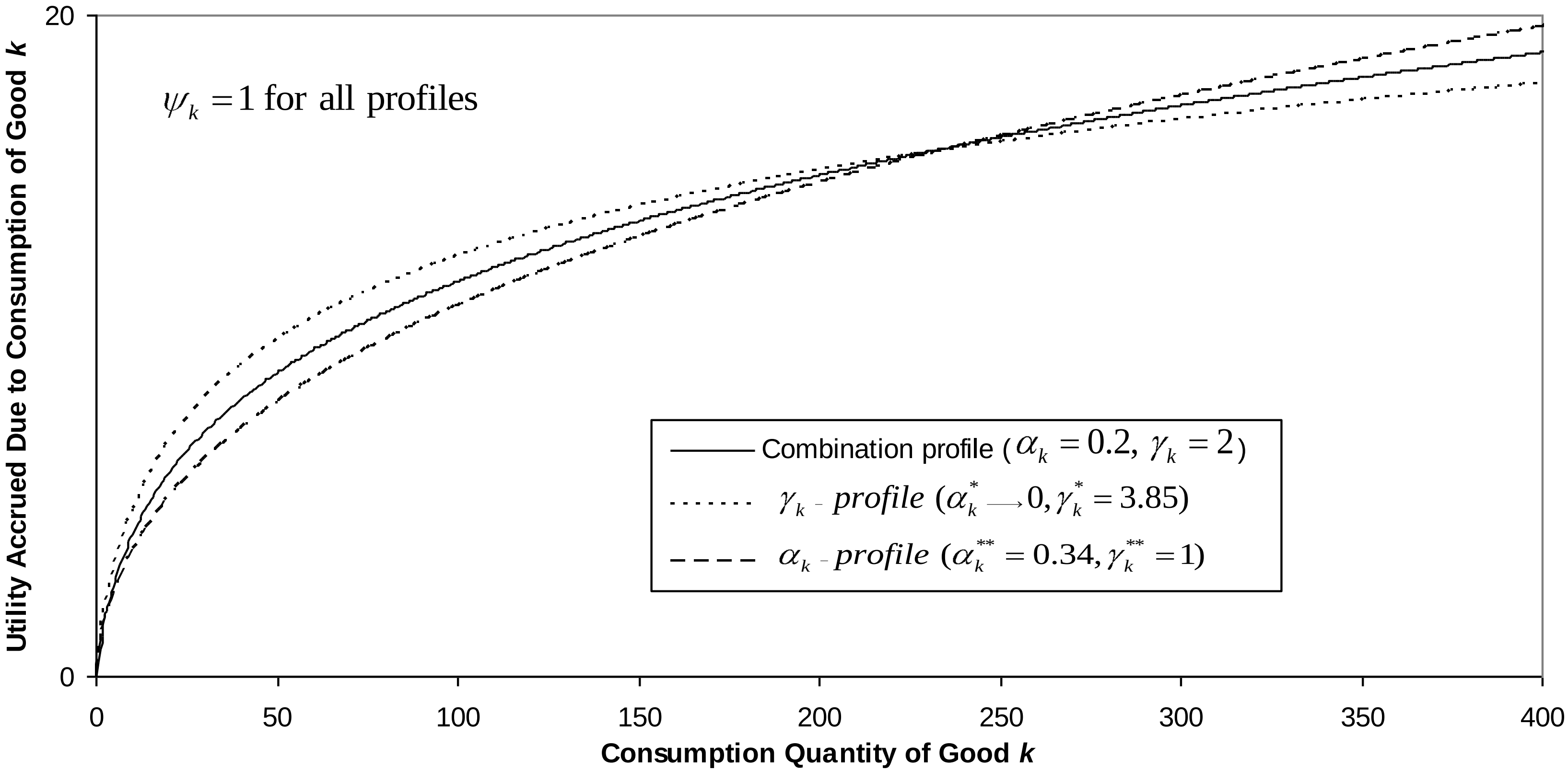
Alternative Profiles for Moderate Satiation Effects with High α_k Value and Low Value

Empirical identification issues associated with utility form-cont'd



Alternative Profiles for Low Satiation Effects with High α_k Value and High Value

Empirical identification issues associated with utility form-cont'd



Alternative Profiles for High Satiation Effects with Low α_k Value and Low Value

Stochastic form of utility function

- Overall random utility function

$$U(\boldsymbol{x}) = \sum_k \frac{\gamma_k}{\alpha_k} \left[\exp(\beta' z_k + \varepsilon_k)\right] \cdot \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\}$$

- Random utility function for optimal expenditure allocations

$$U(\boldsymbol{x}) = \sum_k \frac{\gamma_k}{\alpha_k} \exp(\beta' z_k + \varepsilon_k) \cdot \left\{ \left(\frac{e_k}{\gamma_k p_k} + 1 \right)^{\alpha_k} - 1 \right\}$$

KT conditions

$$V_k + \varepsilon_k = V_1 + \varepsilon_1 \quad \text{if} \quad e_k^* > 0 \qquad (k = 2, 3, \ldots, K)$$

$$V_k + \varepsilon_k < V_1 + \varepsilon_1 \quad \text{if} \quad e_k^* = 0 \qquad (k = 2, 3, \ldots, K), \quad \text{where}$$

$$V_k = \beta' z_k + (\alpha_k - 1)\ln\left(\frac{e_k^*}{\gamma_k p_k} + 1\right) - \ln p_k \quad (k = 1, 2, 3, \ldots, K)$$

General econometric model structure and identification

$$P(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0) = |J| \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_{M+1}=-\infty}^{V_1-V_{M+1}+\varepsilon_1} \int_{\varepsilon_{M+2}=-\infty}^{V_1-V_{M+2}+\varepsilon_1} \cdots \int_{\varepsilon_{K-1}=-\infty}^{V_1-V_{K-1}+\varepsilon_1} \int_{\varepsilon_K=-\infty}^{V_1-V_K+\varepsilon_1}$$
$$f(\varepsilon_1, V_1 - V_2 + \varepsilon_1, V_1 - V_3 + \varepsilon_1, ..., V_1 - V_M + \varepsilon_1, \varepsilon_{M+1}, \varepsilon_{M+2}, ..., \varepsilon_{K-1}, \varepsilon_K)$$
$$d\varepsilon_K d\varepsilon_{K-1} ... d\varepsilon_{M+2} d\varepsilon_{M+1} d\varepsilon_1 ,$$

where J is the Jacobian whose elements are given by (see Bhat, 2005a):

$$J_{ih} = \frac{\partial[V_1 - V_{i+1} + \varepsilon_1]}{\partial e_{h+1}^*} \quad ; i, h = 1, 2, \ldots, M-1$$

$$P(e_1^*, e_2^*, e_3^*, ..., e_M^*, 0, 0, ..., 0) = |J| \int_{\tilde{\varepsilon}_{M+1,1}=-\infty}^{V_1-V_{M+1}} \int_{\tilde{\varepsilon}_{M+2,1}=-\infty}^{V_1-V_{M+2}} \cdots \int_{\tilde{\varepsilon}_{K-1,1}=-\infty}^{V_1-V_{K-1}} \int_{\tilde{\varepsilon}_{K,1}=-\infty}^{V_1-V_K}$$
$$g(V_1 - V_2, V_1 - V_3, ..., V_1 - V_M, \tilde{\varepsilon}_{M+1,1}, \tilde{\varepsilon}_{M+2,1}, ..., \tilde{\varepsilon}_{K,1}) d\tilde{\varepsilon}_{K,1} d\tilde{\varepsilon}_{K-1,1} ... d\tilde{\varepsilon}_{M+1,1}$$

Specific model structures

- The MDCEV model structure

$$P \;\; e_1^*, e_2^*, \; e_3^*, \; ..., \; e_M^*, \; 0, \; 0, \; ..., \; 0$$

$$= \; |J| \int_{\varepsilon_1=-\infty}^{\varepsilon_1=+\infty} \left\{ \left(\prod_{i=2}^{M} \frac{1}{\sigma} \lambda \left[\frac{V_1 - V_i + \varepsilon_1}{\sigma} \right] \right) \right\} \times \left\{ \prod_{s=M+1}^{K} \Lambda \left[\frac{V_1 - V_s + \varepsilon_1}{\sigma} \right] \right\} \frac{1}{\sigma} \lambda \left(\frac{\varepsilon_1}{\sigma} \right) d\varepsilon_1$$

$$|J| = \left(\prod_{i=1}^{M} c_i \right) \left(\sum_{i=1}^{M} \frac{1}{c_i} \right), \text{ where } c_i = \left(\frac{1-\alpha_i}{e_i^* + \gamma_i p_i} \right)$$

$$P \;\; e_1^*, e_2^*, \; e_3^*, \; ..., \; e_M^*, \; 0, \; 0, \; ..., \; 0 \; = \frac{1}{\sigma^{M-1}} \left[\prod_{i=1}^{M} c_i \right] \left[\sum_{i=1}^{M} \frac{1}{c_i} \right] \left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma} \right)^{M}} \right] (M-1)!$$

MDCEV model structure cont'd

- Probability of the consumption pattern of the goods (rather than the expenditure pattern) is

$$P\left(x_1^*, x_2^*, x_3^*, \ldots, x_M^*, 0, 0, \ldots, 0\right)$$

$$= \frac{1}{p_1} \cdot \frac{1}{\sigma^{M-1}} \left[\prod_{i=1}^{M} f_i\right] \left[\sum_{i=1}^{M} \frac{p_i}{f_i}\right] \left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma}\right)^M}\right] (M-1)!,$$

where

$$f_i = \left(\frac{1-\alpha_1}{x_i^* + \gamma_i}\right)$$

MDCEV in an activity-based context

- Growing interest in accommodating joint activity participation across household members

- In conventional discrete choice frameworks, the need to generate mutually exclusive alternatives results in an explosion in choice sets

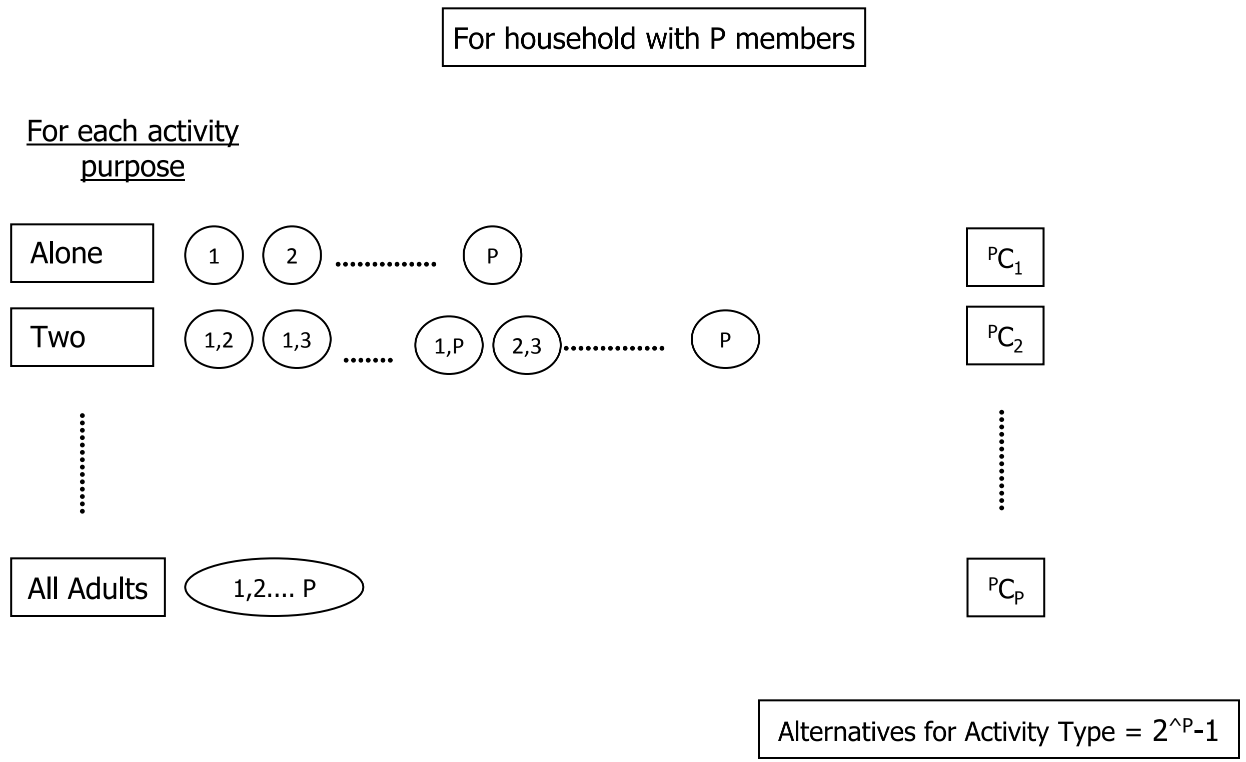
- MDCEV allows us to tackle the problem by considering activity participation as a household decision.

- MDCEV offers substantial computational and behavioral advantages

  - Employ one model to generate activity participation for all household members as opposed to one model per activity type and per person while simultaneously accommodating for joint activity participation

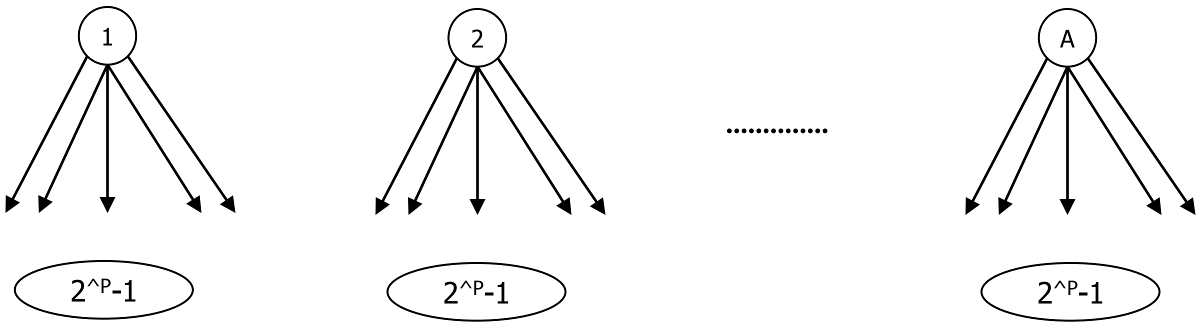
  - Accommodate substitution/complementarity in activity participation and household member dimensions

Activity Generation Framework



MDCEV Framework

<u>Overall choice process</u>
(for A activity purposes)



Total Choice Alternatives = $(2^{\wedge P}-1)(A) + 1$

Traditional Framework

- Within single discrete choice models, there is explosion of alternatives for accommodating joint activities

- We need to determine the entire set of activity purposes pursued and with whom dimension for each of these

Illustration

- For two activity purposes A1 and A2, the possible activity participation:
  - None
  - A1 only
  - A2 only
  - A1 and A2
- So on the activity dimension $2^{\wedge 2} = 2^{\wedge ActNo}$

- For 2 persons P1, P2, the possible combinations:
  - P1 alone
  - P2 alone
  - P1 and P2
- So on the person dimension $2^{\wedge 2}-1 = (2^{\wedge PerNo}-1)$

- For each additional person combination we will have $2^{\wedge ActNo}$ repeating $(2^{\wedge P}-1)$ times
- Now for 2 person and 2 activities we have:
  - $2^{\wedge 2} * 2^{\wedge 2} * 2^{\wedge 2}$

- For A activity purposes, P household members the number of alternatives is given by

$$2^{A} * 2^{A} * 2^{A} \ldots. (2^{P} - 1) \text{ times}$$

$$\boxed{= 2^{A(2^{P}-1)}}$$

Example 2 persons and 2 activity purposes – Single Discrete Case

Each box represents an alternative

P1	P2	P1 P2
None	None	None
A1	None	None
A2	None	None
[illegible]	None	None

P1	P2	P1 P2
None	None	A1
A1	None	A1
A2	None	A1
A1 A2	None	A1

P1	P2	P1 P2
None	None	A2
A1	None	A2
A2	None	A2
A1 A2	None	A2

P1	P2	P1 P2
None	None	A1 A2
A1	None	A1 A2
A2	None	A1 A2
A1 A2	None	A1 A2

[illegible]	P2	P1 P2
None	A1	None
A1	A1	None
A2	A1	None
A1 A2	A1	None

P1	P2	P1 P2
None	A1	A1
A1	A1	A1
A2	A1	A1
A1 A2	A1	A1

P1	P2	P1 P2
None	A1	A2
A1	A1	A2
A2	A1	A2
A1 A2	A1	A2

P1	P2	P1 P2
None	A1	A1 A2
A1	A1	A1 A2
A2	A1	A1 A2
A1 A2	A1	A1 A2

P1	P2	P1 P2
None	A2	None
A1	A2	None
A2	A2	None
A1 A2	A2	None

P1	P2	P1 P2
None	A2	A1
A1	A2	A1
A2	A2	A1
A1 A2	A2	A1

P1	P2	P1 P2
None	A2	A2
A1	A2	A2
A2	A2	A2
A1 A2	A2	A2

P1	P2	P1 P2
None	A2	A1 A2
A1	A2	A1 A2
A2	A2	A1 A2
A1 A2	A2	A1 A2

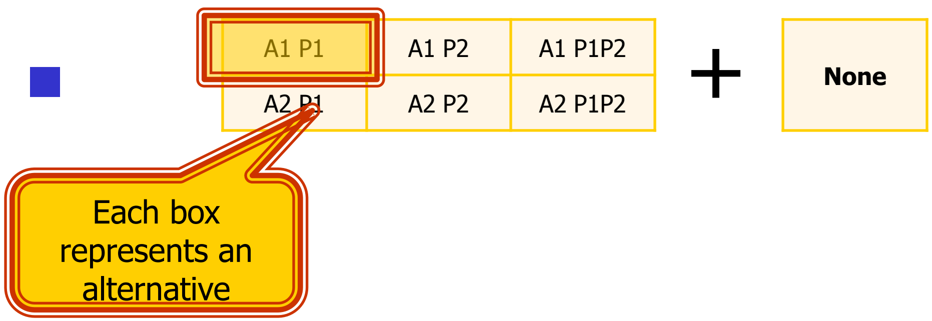
P1	P2	P1 P2
None	A1 A2	None
A1	A1 A2	None
A2	A1 A2	None
A1 A2	A1 A2	None

P1	P2	P1 P2
None	A1 A2	A1
A1	A1 A2	A1
A2	A1 A2	A1
A1 A2	A1 A2	A1

P1	P2	P1 P2
None	A1 A2	A2
A1	A1 A2	A2
A2	A1 A2	A2
A1 A2	A1 A2	A2

P1	P2	P1 P2
None	A1 A2	A1 A2
A1	A1 A2	A1 A2
A2	A1 A2	A1 A2
A1 A2	A1 A2	A1 A2

Example 2 persons and 2 activity purposes – Multiple Discrete Case

- 

- Total 7 alternatives versus 64 in traditional case

Total choice set size comparison for 3 activity purposes

Household Size	Single Discrete Model (MNL)	MDCEV
1	8	3
2	512	9
3	2097152	21
4	3.52×10^{13}	45
5	9.9×10^{27}	93
Total	9.9×10^{27}	171

Once the number of activities increases the difference will be even stark!

MDCEV in Activity-Based Model

- Currently, most activity based models accommodate activity type choice as a series of activity type specific binary logit models for each individual in the household
- These approaches do not explicitly recognize that activity participation is a collective decision of household members
- MDCEV approach, because of its simplicity and relatively inexpensive computational requirement, facilitates modeling activity participation at a household level with joint activity participation incorporated in a simple fashion
- CEMDAP (within SimAGENT) now features MDCEV for activity participation

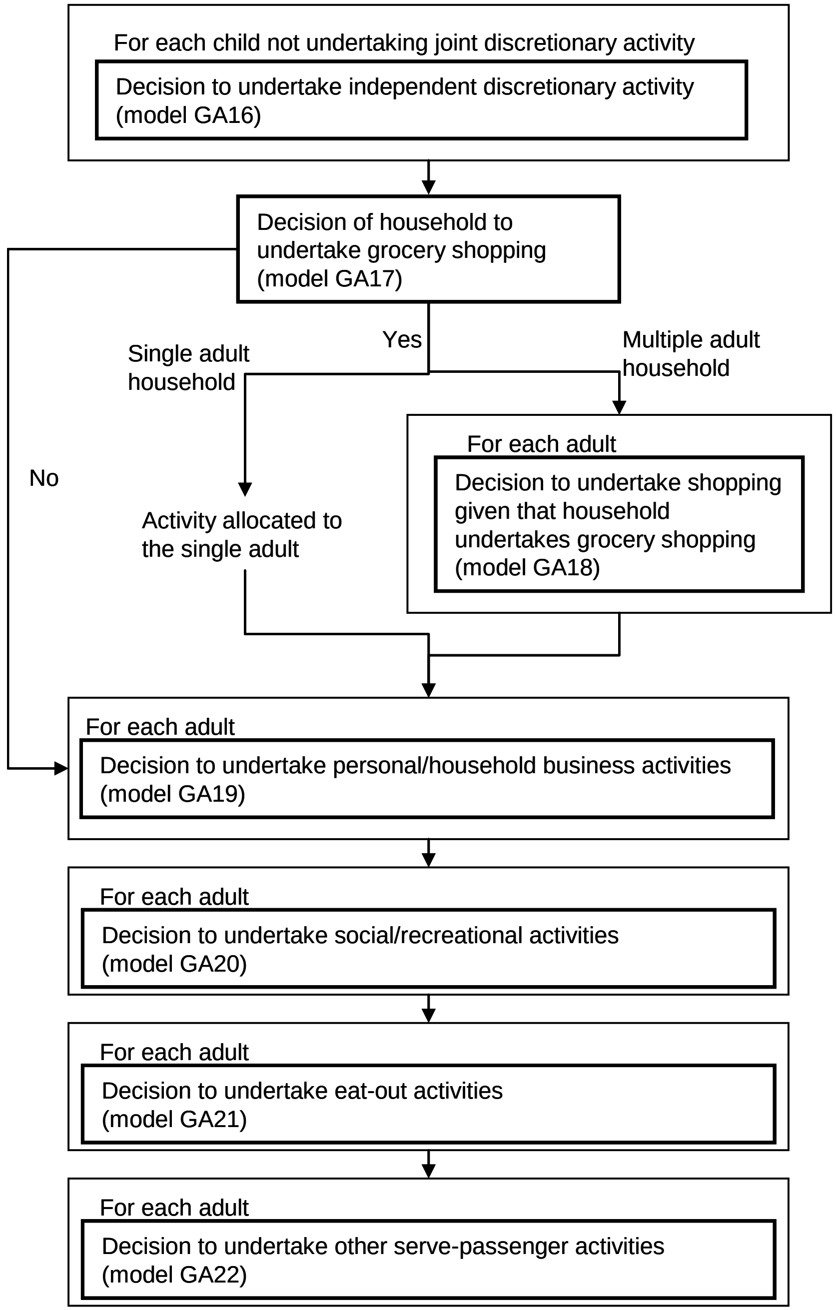
Comparison of some models

Model / Aspect	SF-CHAMP	SACSIM	MORPC	CEMDAP	SimAGENT
MPO	San Francisco County Transportation Authority	Sacramento Area Council of Governments	Mid-Ohio Regional Planning Commission	North Central Texas Council of Governments	Southern California Association of Governments (SCAG)
Region	San Francisco County, CA	Sacramento, CA	Columbus, Ohio	Dallas Fort-Worth, TX	Los Angeles, CA
Base year	1998	2000	2000	2000	2003
Population in base year	0.3 Million Households 0.8 Million Individuals	0.7 Million Households 1.8 Million Individuals	0.6 Million Households 1.4 Million Individuals	1.8 Million Households 4.8 Million Individuals	5.6 Million Households 17.6 Million Individuals

Data: Summary of Reviewed Models

Model Aspect	SF-CHAMP	SACSIM	MORPC	CEMDAP	SimAGENT
Model estimation data	1990 SF Bay Area Household Travel Survey Data of 1100 HHs on SF County, stated preference survey of 609HHs for transit related Preferences	Household activity diary survey	1999 Household travel survey data of 5500 HHs in the Columbus region, on-board transit survey data	1996 Household travel survey data of 3500 households in DFW, on-board transit survey data	California Department of Finance (DOF) E-5 Population and Housing Estimates; California Employment Development Department (EDD) 2005 Benchmark
Network zones (TAZs)	1,900	1,300 (as well as parcels)	2,000 (w/ 3 transit access zones in each zone)	4784	4192 (as well as parcels)
Network time periods	5 per day	4 per day	5 per day	5 per day	4 per day
Predicted time periods	5 per day	30 min	1 hour	continuous time (1 min)	continuous time (1 min)

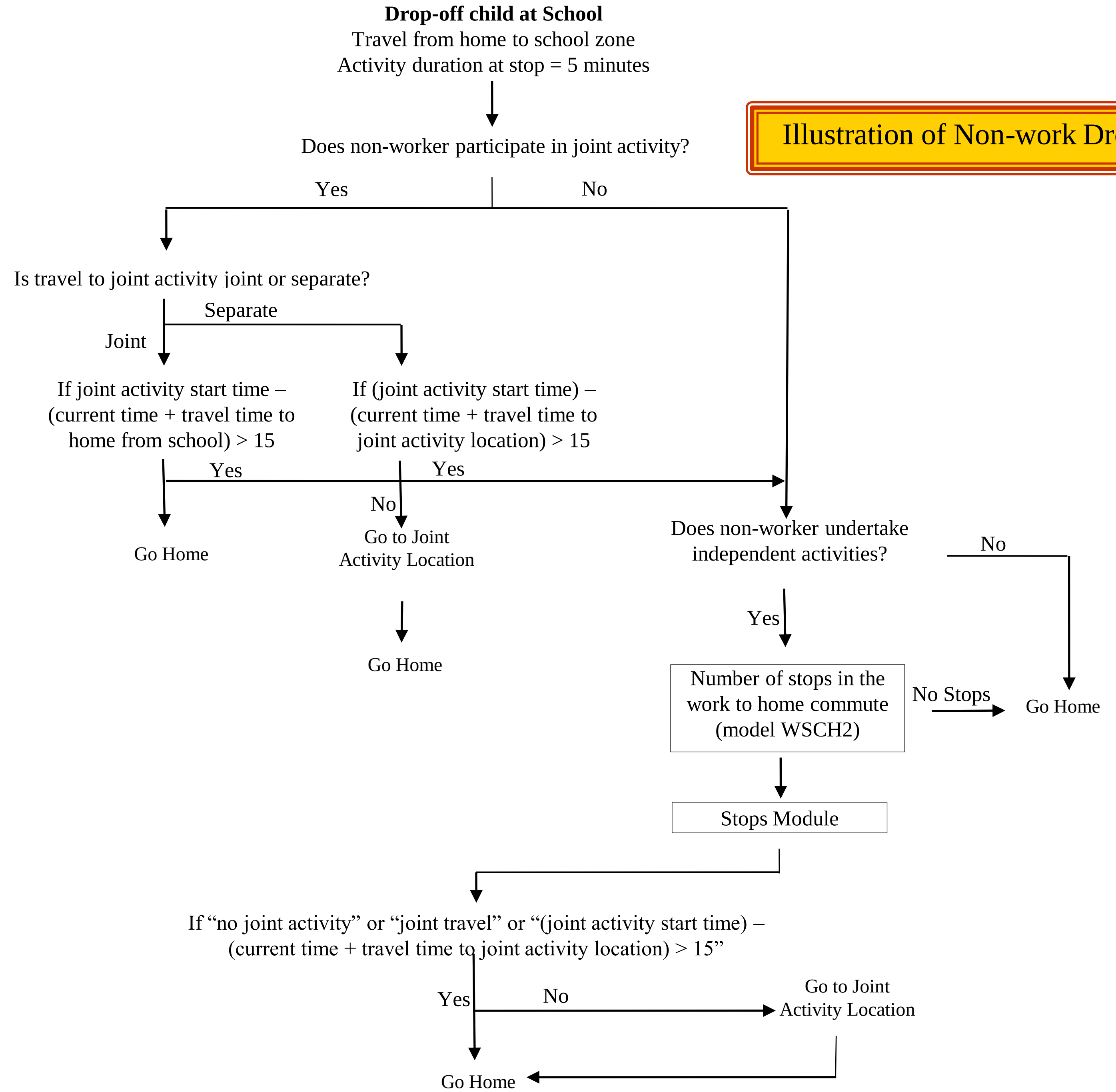
How MDCEV alters CEMDAP



For every household model activity participation using MDCEV Model

- Incorporating joint activity alters travel scheduling process substantially
- MDCEV provides us the household members for joint activity
- Need to ensure spatial and temporal consistency among the joint activity participants
- Determining when and where the joint activity is pursued forms an additional pin around which individual travel is scheduled
- Currently we follow the following precedence
  - Children travel needs
  - Commuter travel needs
  - Joint travel (excluding children travel needs)
  - Individual travel

Illustration of Non-work Drop-off tour



Summary

- Several multiple discrete continuous choice contexts can be modeled using MDCEV
- MDCEV is an effective tool to address the computational and behavioral challenges to model activity participation (while incorporating joint activity participation seamlessly)
- The computational advantages are evident based on the numbers provided
- CEMDAP (within SimAGENT), in its latest version, will feature MDCEV

A NEW ESTIMATION APPROACH FOR DISCRETE CHOICE MODELING SYSTEMS

Motivation

- Simulation techniques:
  - Maximum Simulated Likelihood (MSL) Approach
    - *Approach gets imprecise, develops convergence problems, and becomes computationally expensive with the increase in the number of ordered-response outcomes*
  - Bayesian Inference Approach
    - *Unfortunately, the method remains cumbersome, requires extensive simulation, and is time-consuming*
  - Overall, simulation-based approaches become impractical or even infeasible as the number of ordered-response outcomes increases

Motivation

- Another solution to such problems is the use of the Composite Marginal Likelihood (CML) approach. The CML approach…
  - Belongs to the more general class of composite likelihood function approaches
  - Is based on forming a surrogate likelihood function that compounds much easier-to-compute, lower-dimensional, marginal likelihoods
  - Represents a conceptually and pedagogically simpler simulation-free procedure relative to simulation techniques
  - Can be applied using simple optimization software for likelihood estimation
  - Is typically more robust, and has the advantage of reproducibility of the results

Motivation

- We demonstrate the use of the CML approach in a pairwise marginal likelihood setting
  - The pairwise marginal likelihood is formed by the product of likelihood contributions of all subset of couplets (i.e., pairs of variables or pairs of observations)
  - Under the usual regularity assumptions, the CML (and hence, the pairwise marginal likelihood) estimator is consistent and asymptotically normal distributed
  - This is because of the unbiasedness of the CML score function, which is a linear combination of proper score functions associated with the marginal event probabilities forming the composite likelihood

Motivation

- Compare the performance of the MSL approach with the CML approach when the MSL approach is feasible
- Undertake a comparison in the context of an ordered-response setting with different correlation structures and with:
  - cross-sectional data, and
  - panel data
- Use simulated data sets to evaluate the two estimation approaches
- Examine the performance of the MSL and CML approaches in terms of:
  - The ability of the two approaches to recover model parameters
  - Relative efficiency, and
  - Non-convergence and computational cost

Focus on Ordered Response Systems

- Ordered response model systems are used when analyzing ordinal discrete outcome data that may be considered as manifestations of an underlying scale that is endowed with a natural ordering
- Examples include
  - Ratings data (of consumer products, bonds, credit evaluation, movies, *etc.*)
  - Likert-scale type attitudinal/opinion data (of air pollution levels, traffic congestion levels, school academic curriculum satisfaction levels, teacher evaluations, *etc.*)
  - Grouped data (such as bracketed income data in surveys or discretized rainfall data)
  - Count data (such as the number of trips made by a household, the number of episodes of physical activity pursued by an individual, and the number of cars owned by a household)

Focus on Ordered Response Systems

- There is an abundance of applications of the ordered-response model in the literature
  - Examples include applications in the sociological, biological, marketing, and transportation sciences

- Mostly one outcome variable, though there have been some applications with 2 to 3 outcome variables

- However, the examination of more than three correlated outcomes is rare because of difficulty associated with medium-to-high dimensional integration

Focus on Ordered Response Systems

- Cross-sectional examples of multiple outcome variables
  - Number of episodes of each of several activities
  - Satisfaction levels associated with a related set of products/services
  - Multiple ratings measures regarding the state of health of an individual/organization

- Time-series or panel examples of multiple outcome variables
  - Rainfall levels (measured in grouped categories) over time in each of several spatial regions
  - Individual stop-making behavior over multiple days in a week
  - Individual headache severity levels at different points in time

Econometric Framework

Multivariate Ordered-Response Model System - Cross-Sectional Formulation (CMOP Model)

$$y_{qi}^{*} = \beta_i' x_{qi} + \varepsilon_{qi}, y_{qi} = m_{qi} \text{ if } \theta_i^{m_{qi}-1} < y_{qi}^{*} < \theta_i^{m_{qi}}$$

y_{qi}^{*} = The latent propensity of individual q for ordered-response variable i

x_{qi} = A $(L \times 1)$ vector of exogenous variables (not including a constant)

β_i = A corresponding $(L \times 1)$ vector of coefficients (to be estimated)

ε_{qi} = A standard normal error term,

y_{qi} = "Observed" count value for individual q for variable i

$\theta_i^{m_{qi}-1}$ = The lower bound threshold for discrete level m_{qi} of variable i, and

$\theta_i^{m_{qi}}$ = The upper bound threshold for discrete level m_{qi} of variable i

Econometric Framework

$$\varepsilon_q \sim N\left[\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1I}\\ \rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2I}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \rho_{I1} & \rho_{I2} & \rho_{I3} & \cdots & 1\end{pmatrix}\right]$$

Econometric Framework

$$\delta = (\beta_1', \beta_2', ..., \beta_I';\ \theta_1', \theta_2', ..., \theta_I';\ \Omega')',$$

$$\theta_i = (\theta_i^1, \theta_i^2, ..., \theta_i^{K_i})'$$

$$L_q(\delta) = \Pr(y_{q1} = m_{q1}, y_{q2} = m_{q2}, ..., y_{qI} = m_{qI})$$

$$L_q(\delta) = \int_{v_1=\theta_1^{m_{q1}} - \beta_1' x_{q1}}^{\theta_1^{m_{q1}+1} - \beta_1' x_{q1}} \int_{v_2=\theta_2^{m_{q2}} - \beta_2' x_{q2}}^{\theta_2^{m_{q2}+1} - \beta_2' x_{q2}} \cdots \int_{v_I=\theta_I^{m_{qI}} - \beta_I' x_{qI}}^{\theta_I^{m_{qI}+1} - \beta_I' x_{qI}} \phi_I(v_1, v_2, ..., v_I \mid \Omega) dv_1 dv_2 ... dv_I$$

Econometric Framework

Multivariate Ordered-Response Model System - Panel Formulation (PMOP Model)

$$y_{qj}^{*} = \beta' x_{qj} + u_q + \varepsilon_{qj}, y_{qj} = m_{qj} \text{ if } \theta^{m_{qj}-1} < y_{qj}^{*} < \theta^{m_{qj}}$$

y_{qj}^{*} = The latent propensity of individual q for jth observation

β = A corresponding $(L \times 1)$ vector of coefficients (to be estimated)

x_{qj} = A $(L \times 1)$ vector of exogenous variables (not including a constant)

u_q = An individual-specific random term, $u_q \overset{i.i.d}{\sim} N(0, \sigma^2)$

ε_{qj} = A standard normal error term uncorrelated across individuals q, but serially correlated across observations j for individual q

y_{qj} = "Observed" count value for individual q for variable j

$\theta^{m_{qj}-1}$ = The lower bound threshold for discrete level m_{qj}

$\theta^{m_{qj}}$ = The upper bound threshold for discrete level m_{qj}

Econometric Framework

The joint distribution of the latent variables $(y^*_{q1}, y^*_{q2}, \ldots y^*_{qJ})$ for the *q*th subject is multivariate normal with standardized mean vector $(\beta' x_{q1}/\mu, \beta' x_{q2}/\mu, \ldots \beta' x_{qJ}/\mu)$ and a correlation matrix with constant non-diagonal entries σ^2/μ^2, where $\mu = \sqrt{1+\sigma^2}$

$$\delta = (\beta';\ \theta^1, \theta^2, \ldots, \theta^{K-1};\ \sigma, \rho)'$$

$$L_q(\delta) = \Pr(y_{q1} = m_{q1}, y_{q2} = m_{q2}, \ldots, y_{qJ} = m_{qJ})$$

$$L_q(\delta) = \int_{v_1=\alpha^{m_{q1}-1}}^{\alpha^{m_{q1}}} \int_{v_2=\alpha^{m_{q2}-1}}^{\alpha^{m_{q2}}} \cdots \int_{v_J=\alpha^{m_{qJ}-1}}^{\alpha^{m_{qJ}}} \phi_J(v_1, v_2, \ldots, v_J \mid R_q)\, dv_1 dv_2 \ldots dv_J$$

where $\alpha^{m_{qj}} = (\theta^{m_{qj}} - \beta' x_{qj})/\mu.$

Estimation Approaches

- **Simulation Approaches**
  - The Frequentist Approach – Maximum Simulated Likelihood (MSL) Method
  - The Bayesian Approach
  - Simulators Used in the Current Paper
    - *The GHK Probability Simulator for the CMOP Model*
    - *The GB Simulator for the PMOP Model*

- **The CML Technique – The Pairwise Marginal Likelihood Inference Approach**
  - Pairwise Likelihood Approach
    - *The CMOP Model*
    - *The PMOP Model*
    - *Positive-Definiteness of the Correlation Matrix*

Estimation Approaches

The GHK Probability Simulator for the CMOP Model

- Named after John F. Geweke, Vassilis A. Hajivassiliou, and Michael P. Keane

- The GHK is perhaps the most widely used probability simulator for integration of the multivariate normal density function

- The simulator is based on directly approximating the probability of a multivariate rectangular region of the multivariate normal density distribution

The GHK Probability Simulator for the CMOP Model (cont.)

$$L_q(\delta) = \Pr(y_{q1} = m_{q1}, y_{q2} = m_{q2}, \ldots, y_{qJ} = m_{qJ})$$

$$L_q(\delta) = \Pr(y_{q1} = m_{q1}) \Pr(y_{q2} = m_{q2} \mid y_{q1} = m_{q1}) \Pr(y_{q3} = m_{q3} \mid y_{q1} = m_{q1}, y_{q2} = m_{q2}) \ldots$$

$$\ldots \Pr(y_{qI} = m_{qI} \mid y_{q1} = m_{q1}, y_{q2} = m_{q2}, \ldots, y_{qI-1} = m_{qI-1})$$

Also,,

$$\begin{bmatrix} \varepsilon_{q1} \\ \varepsilon_{q2} \\ \vdots \\ \varepsilon_{qI} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{I1} & l_{I2} & l_{I3} & \cdots & l_{II} \end{bmatrix} \begin{bmatrix} v_{q1} \\ v_{q2} \\ \vdots \\ v_{qI} \end{bmatrix}$$

$$\varepsilon_q = \boldsymbol{L} v_q$$

Estimation Approaches

The GHK Probability Simulator for the CMOP Model (cont.)

- $\boldsymbol{L}$ is the lower triangular Cholesky decomposition of the correlation matrix $\boldsymbol{\Sigma}$, and v_q terms are independent and identically distributed standard normal deviates

- V_q are drawn d times (d = 1, 2, …, 100) from the univariate standard normal distribution with pre-specified lower and upper bounds

- We use a randomized Halton draw procedure to generate the d realizations

- The positive definiteness of $\boldsymbol{\Sigma}$ was ensured by parameterizing the likelihood function with the elements of $\boldsymbol{L}$

Estimation Approaches

The GB Simulator for the PMOP Model

- Named after Alan Genz and Frank Bretz
- Provides an alternative simulation-based approximation of multivariate normal probabilities
- The approach involves
  - Transforming the original hyper-rectangle integral region to an integral over a unit hypercube
  - Filling the transformed integral region by randomized lattice points
  - Deriving robust integration error bounds by means of additional shifts of the integration nodes in random directions
- The positive-definite correlation matrix is ensured by defining the parameter spaces, so that $\sigma > 0$ and $0 < \rho < 1$

Estimation Approaches

Pairwise Likelihood Approach for the CMOP Model

$$L_{CML,q}^{CMOP}(\delta) = \prod_{i=1}^{I-1} \prod_{g=i+1}^{I} \Pr(y_{qi} = m_{qi}, y_{qg} = m_{qg})$$

$$= \prod_{i=1}^{I-1} \prod_{g=i+1}^{I} \left[\begin{array}{l} \Phi_2\left(\theta_i^{m_{qi}} - \beta_i' x_{qi}, \theta_g^{m_{qg}} - \beta_g' x_{qg}, \rho_{ig}\right) - \Phi_2\left(\theta_i^{m_{qi}} - \beta_i' x_{qi}, \theta_g^{m_{qg}-1} - \beta_g' x_{qg}, \rho_{ig}\right) \\ -\Phi_2\left(\theta_i^{m_{qi}-1} - \beta_i' x_{qi}, \theta_g^{m_{qg}} - \beta_g' x_{qg}, \rho_{ig}\right) + \Phi_2\left(\theta_i^{m_{qi}-1} - \beta_i' x_{qi}, \theta_g^{m_{qg}-1} - \beta_g' x_{qg}, \rho_{ig}\right) \end{array} \right]$$

$$L_{CML}^{CMOP}(\delta) = \prod_{q} L_{CML,q}^{CMOP}(\delta)$$

Estimation Approaches

Pairwise Likelihood Approach for the PMOP Model

$$L_{CML,q}^{PMOP}(\delta) = \prod_{j=1}^{J-1} \prod_{g=j+1}^{J} \Pr(y_{qj} = m_{qj}, y_{qg} = m_{qg})$$

$$= \prod_{j=1}^{J-1} \prod_{g=j+1}^{J} \left[\begin{array}{l} \Phi_2\left(\alpha^{m_{qj}}, \alpha^{m_{qg}}, \rho_{jg}\right) - \Phi_2\left(\alpha^{m_{qj}}, \alpha^{m_{qg-1}}, \rho_{jg}\right) \\ -\Phi_2\left(\alpha^{m_{qj}-1}, \alpha^{m_{qg}}, \rho_{jg}\right) + \Phi_2\left(\alpha^{m_{qj}-1}, \alpha^{m_{qg}-1}, \rho_{jg}\right) \end{array} \right],$$

$$L_{CML}^{PMOP}(\delta) = \prod_{q} L_{CML,q}^{PMOP}(\delta)$$

Where,

$$\alpha^{m_{qj}} = (\theta^{m_{qj}} - \beta' x_{qj}) / \mu, \;\; \mu = \sqrt{1+\sigma^2}, \;\; \text{and} \;\; \rho_{jg} = (\sigma^2 + \rho^{|t_{qj} - t_{qg}|}) / \mu^2$$

Experimental Design

The CMOP Model

- Multivariate ordered response system with five ordinal variables
  - Low error correlation structure (Σ_{low})
  - High error correlation structure (Σ_{high})
- δ vector with pre-specified values
- 20 independent data sets with 1000 data points
  - The GHK simulator is applied to each data set
    - *Using 100 draws per individual of the randomized Halton sequence*
    - *10 times with different (independent) randomized Halton draw sequences*

Experimental Design

The PMOP Model

- Multivariate ordered response system with six ordinal variables
  - Low autoregressive correlation parameter (ρ=0.3)
  - High autoregressive correlation parameter (ρ=0.7)

- δ vector with pre-specified values

- 100 independent data sets with 200 subjects and 6 "observation" per subject
  - The GB simulator is applied to each data set
    - *10 times with different (independent) random draw sequences*
    - *With an absolute error tolerance of 0.001*

Performance Measures

- Parameter Estimates
  - Mean estimate
  - Absolute percentage bias
- Standard Error Estimates
  - Finite sample standard error
  - Asymptotic standard error
  - In addition, for MSL approach we estimated:
    - Simulation standard error
    - Simulation adjusted standard error

Performance Measures

- Relative Efficiencies
  - Ratio between the MSL and CML asymptotic standard errors
  - Ratio between the simulation adjusted standard error and the CML asymptotic standard error
- Non-convergence Rates
- Relative Computational Time Factor (RCTF)

S

Parameter	True Value	MSL Approach						CML Approach				Rel. Eff.	
		Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates		$\frac{A}{C}$	$\frac{B}{C}$
		Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (A)	Simulation SE	Simulation Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (C)		
Coefficients													
β_{11}	0.5000	0.5167	3.34%	0.0481	0.0399	0.0014	0.0399	0.5021	0.43%	0.0448	0.0395	1.0109	1.0116
β_{21}	1.0000	1.0077	0.77%	0.0474	0.0492	0.0005	0.0492	1.0108	1.08%	0.0484	0.0482	1.0221	1.0222
β_{31}	0.2500	0.2501	0.06%	0.0445	0.0416	0.0010	0.0416	0.2568	2.73%	0.0252	0.0380	1.0957	1.0961
β_{12}	0.7500	0.7461	0.52%	0.0641	0.0501	0.0037	0.0503	0.7698	2.65%	0.0484	0.0487	1.0283	1.0311
β_{22}	1.0000	0.9984	0.16%	0.0477	0.0550	0.0015	0.0550	0.9990	0.10%	0.0503	0.0544	1.0100	1.0104
β_{32}	0.5000	0.4884	2.31%	0.0413	0.0433	0.0017	0.0434	0.5060	1.19%	0.0326	0.0455	0.9518	0.9526
β_{42}	0.2500	0.2605	4.19%	0.0372	0.0432	0.0006	0.0432	0.2582	3.30%	0.0363	0.0426	1.0149	1.0150
β_{13}	0.2500	0.2445	2.21%	0.0401	0.0346	0.0008	0.0346	0.2510	0.40%	0.0305	0.0342	1.0101	1.0104
β_{23}	0.5000	0.4967	0.66%	0.0420	0.0357	0.0021	0.0358	0.5063	1.25%	0.0337	0.0364	0.9815	0.9833
β_{33}	0.7500	0.7526	0.34%	0.0348	0.0386	0.0005	0.0386	0.7454	0.62%	0.0441	0.0389	0.9929	0.9930
β_{14}	0.7500	0.7593	1.24%	0.0530	0.0583	0.0008	0.0583	0.7562	0.83%	0.0600	0.0573	1.0183	1.0184
β_{24}	0.2500	0.2536	1.46%	0.0420	0.0486	0.0024	0.0487	0.2472	1.11%	0.0491	0.0483	1.0067	1.0079
β_{34}	1.0000	0.9976	0.24%	0.0832	0.0652	0.0017	0.0652	1.0131	1.31%	0.0643	0.0633	1.0298	1.0301
β_{44}	0.3000	0.2898	3.39%	0.0481	0.0508	0.0022	0.0508	0.3144	4.82%	0.0551	0.0498	1.0199	1.0208
β_{15}	0.4000	0.3946	1.34%	0.0333	0.0382	0.0014	0.0382	0.4097	2.42%	0.0300	0.0380	1.0055	1.0061
β_{25}	1.0000	0.9911	0.89%	0.0434	0.0475	0.0016	0.0475	0.9902	0.98%	0.0441	0.0458	1.0352	1.0358
β_{35}	0.6000	0.5987	0.22%	0.0322	0.0402	0.0007	0.0402	0.5898	1.69%	0.0407	0.0404	0.9959	0.9961

Simulation Results – The CMOP Model (Σ_{low})

Para-meter	True Value	MSL Approach						CML Approach				Rel. Eff.	
		Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates		$\frac{A}{C}$	$\frac{B}{C}$
		Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (A)	Simula-tion SE	Simula-tion Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (C)		
Correlation Coefficients													
ρ_{12}	0.3000	0.2857	4.76%	0.0496	0.0476	0.0020	0.0476	0.2977	0.77%	0.0591	0.0467	1.0174	1.0184
ρ_{13}	0.2000	0.2013	0.66%	0.0477	0.0409	0.0019	0.0410	0.2091	4.56%	0.0318	0.0401	1.0220	1.0231
ρ_{14}	0.2200	0.1919	12.76%	0.0535	0.0597	0.0035	0.0598	0.2313	5.13%	0.0636	0.0560	1.0664	1.0682
ρ_{15}	0.1500	0.1739	15.95%	0.0388	0.0439	0.0040	0.0441	0.1439	4.05%	0.0419	0.0431	1.0198	1.0239
ρ_{23}	0.2500	0.2414	3.46%	0.0546	0.0443	0.0040	0.0445	0.2523	0.92%	0.0408	0.0439	1.0092	1.0133
ρ_{24}	0.3000	0.2960	1.34%	0.0619	0.0631	0.0047	0.0633	0.3013	0.45%	0.0736	0.0610	1.0342	1.0372
ρ_{25}	0.1200	0.1117	6.94%	0.0676	0.0489	0.0044	0.0491	0.1348	12.34%	0.0581	0.0481	1.0154	1.0194
ρ_{34}	0.2700	0.2737	1.37%	0.0488	0.0515	0.0029	0.0516	0.2584	4.28%	0.0580	0.0510	1.0094	1.0110
ρ_{35}	0.2000	0.2052	2.62%	0.0434	0.0378	0.0022	0.0378	0.1936	3.22%	0.0438	0.0391	0.9662	0.9678
ρ_{45}	0.2500	0.2419	3.25%	0.0465	0.0533	0.0075	0.0538	0.2570	2.78%	0.0455	0.0536	0.9937	1.0034

Simulation Results – The CMOP Model (Σ_{low})

Para-meter	True Value	MSL Approach						CML Approach				Rel. Eff.	
		Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates		$\frac{A}{C}$	$\frac{B}{C}$
		Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (A)	Simula-tion SE	Simula-tion Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (C)		
Threshold Parameters													
θ_1^1	-1.0000	-1.0172	1.72%	0.0587	0.0555	0.0007	0.0555	-1.0289	2.89%	0.0741	0.0561	0.9892	0.9893
θ_1^2	1.0000	0.9985	0.15%	0.0661	0.0554	0.0011	0.0554	1.0010	0.10%	0.0536	0.0551	1.0063	1.0065
θ_1^3	3.0000	2.9992	0.03%	0.0948	0.1285	0.0034	0.1285	2.9685	1.05%	0.1439	0.1250	1.0279	1.0282
θ_2^1	0.0000	-0.0172	-	0.0358	0.0481	0.0007	0.0481	-0.0015	-	0.0475	0.0493	0.9750	0.9751
θ_2^2	2.0000	1.9935	0.32%	0.0806	0.0831	0.0030	0.0831	2.0150	0.75%	0.0904	0.0850	0.9778	0.9784
θ_3^1	-2.0000	-2.0193	0.97%	0.0848	0.0781	0.0019	0.0781	-2.0238	1.19%	0.0892	0.0787	0.9920	0.9923
θ_3^2	-0.5000	-0.5173	3.47%	0.0464	0.0462	0.0005	0.0462	-0.4968	0.64%	0.0519	0.0465	0.9928	0.9928
θ_3^3	1.0000	0.9956	0.44%	0.0460	0.0516	0.0011	0.0516	1.0014	0.14%	0.0584	0.0523	0.9877	0.9879
θ_3^4	2.5000	2.4871	0.52%	0.0883	0.0981	0.0040	0.0982	2.5111	0.44%	0.0735	0.1002	0.9788	0.9796
θ_4^1	1.0000	0.9908	0.92%	0.0611	0.0615	0.0031	0.0616	1.0105	1.05%	0.0623	0.0625	0.9838	0.9851
θ_4^2	3.0000	3.0135	0.45%	0.1625	0.1395	0.0039	0.1396	2.9999	0.00%	0.1134	0.1347	1.0356	1.0360
θ_5^1	-1.5000	-1.5084	0.56%	0.0596	0.0651	0.0032	0.0652	-1.4805	1.30%	0.0821	0.0656	0.9925	0.9937
θ_5^2	0.5000	0.4925	1.50%	0.0504	0.0491	0.0017	0.0492	0.5072	1.44%	0.0380	0.0497	0.9897	0.9903
θ_5^3	2.0000	2.0201	1.01%	0.0899	0.0797	0.0017	0.0798	2.0049	0.24%	0.0722	0.0786	1.0151	1.0154
Overall mean value across parameters		-	2.21%	0.0566	0.0564	0.0022	0.0564	-	1.92%	0.0562	0.0559	1.0080	1.0092

Simulation Results – The CMOP Model (Σ_{high})

Para-meter	True Value	MSL Approach						CML Approach				Rel. Eff.	
		Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates		$\frac{A}{C}$	$\frac{B}{C}$
		Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (A)	Simula-tion SE	Simula-tion Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymp-totic SE (C)		
Coefficients													
β_{11}	0.5000	0.5063	1.27%	0.0300	0.0294	0.0020	0.0294	0.5027	0.54%	0.0292	0.0317	0.9274	0.9294
β_{21}	1.0000	1.0089	0.89%	0.0410	0.0391	0.0026	0.0392	1.0087	0.87%	0.0479	0.0410	0.9538	0.9560
β_{31}	0.2500	0.2571	2.85%	0.0215	0.0288	0.0017	0.0289	0.2489	0.42%	0.0251	0.0290	0.9943	0.9961
β_{12}	0.7500	0.7596	1.27%	0.0495	0.0373	0.0028	0.0374	0.7699	2.65%	0.0396	0.0395	0.9451	0.9477
β_{22}	1.0000	1.0184	1.84%	0.0439	0.0436	0.0036	0.0437	1.0295	2.95%	0.0497	0.0463	0.9419	0.9451
β_{32}	0.5000	0.5009	0.17%	0.0343	0.0314	0.0023	0.0315	0.5220	4.39%	0.0282	0.0352	0.8931	0.8955
β_{42}	0.2500	0.2524	0.96%	0.0284	0.0294	0.0021	0.0294	0.2658	6.34%	0.0263	0.0315	0.9318	0.9343
β_{13}	0.2500	0.2473	1.08%	0.0244	0.0233	0.0015	0.0234	0.2605	4.18%	0.0269	0.0251	0.9274	0.9293
β_{23}	0.5000	0.5084	1.67%	0.0273	0.0256	0.0020	0.0256	0.5100	2.01%	0.0300	0.0277	0.9221	0.9248
β_{33}	0.7500	0.7498	0.02%	0.0302	0.0291	0.0019	0.0291	0.7572	0.96%	0.0365	0.0318	0.9150	0.9170
β_{14}	0.7500	0.7508	0.11%	0.0416	0.0419	0.0039	0.0420	0.7707	2.75%	0.0452	0.0450	0.9302	0.9341
β_{24}	0.2500	0.2407	3.70%	0.0311	0.0326	0.0033	0.0327	0.2480	0.80%	0.0234	0.0363	0.8977	0.9022
β_{34}	1.0000	1.0160	1.60%	0.0483	0.0489	0.0041	0.0491	1.0000	0.00%	0.0360	0.0513	0.9532	0.9566
β_{44}	0.3000	0.3172	5.72%	0.0481	0.0336	0.0028	0.0337	0.3049	1.62%	0.0423	0.0368	0.9133	0.9165
β_{15}	0.4000	0.3899	2.54%	0.0279	0.0286	0.0026	0.0288	0.4036	0.90%	0.0274	0.0301	0.9516	0.9554
β_{25}	1.0000	0.9875	1.25%	0.0365	0.0391	0.0036	0.0393	1.0008	0.08%	0.0452	0.0398	0.9821	0.9862
β_{35}	0.6000	0.5923	1.28%	0.0309	0.0316	0.0030	0.0317	0.6027	0.45%	0.0332	0.0329	0.9607	0.9649

Simulation Results – The CMOP Model (Σ_{high})

Parameter	True Value	MSL Approach						CML Approach				Rel. Eff.	
		Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates			
		Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (A)	Simulation SE	Simulation Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (C)	$\frac{A}{C}$	$\frac{B}{C}$
Correlation Coefficients													
ρ_{12}	0.9000	0.8969	0.34%	0.0224	0.0177	0.0034	0.0180	0.9019	0.21%	0.0233	0.0183	0.9669	0.9845
ρ_{13}	0.8000	0.8041	0.51%	0.0174	0.0201	0.0035	0.0204	0.8009	0.11%	0.0195	0.0203	0.9874	1.0023
ρ_{14}	0.8200	0.8249	0.60%	0.0284	0.0265	0.0061	0.0272	0.8151	0.60%	0.0296	0.0297	0.8933	0.9165
ρ_{15}	0.7500	0.7536	0.49%	0.0248	0.0243	0.0046	0.0247	0.7501	0.01%	0.0242	0.0251	0.9678	0.9849
ρ_{23}	0.8500	0.8426	0.87%	0.0181	0.0190	0.0081	0.0207	0.8468	0.38%	0.0190	0.0198	0.9606	1.0438
ρ_{24}	0.9000	0.8842	1.75%	0.0187	0.0231	0.0097	0.0251	0.9023	0.26%	0.0289	0.0244	0.9484	1.0284
ρ_{25}	0.7200	0.7184	0.22%	0.0241	0.0280	0.0072	0.0289	0.7207	0.09%	0.0295	0.0301	0.9298	0.9600
ρ_{34}	0.8700	0.8724	0.27%	0.0176	0.0197	0.0036	0.0200	0.8644	0.65%	0.0208	0.0220	0.8972	0.9124
ρ_{35}	0.8000	0.7997	0.04%	0.0265	0.0191	0.0039	0.0195	0.7988	0.15%	0.0193	0.0198	0.9645	0.9848
ρ_{45}	0.8500	0.8421	0.93%	0.0242	0.0231	0.0128	0.0264	0.8576	0.89%	0.0192	0.0252	0.9156	1.0480

Simulation Results – The CMOP Model (Σ_{high})

Parameter	True Value	Parameter Estimates		Standard Error (SE) Estimates				Parameter Estimates		Standard Error (SE) Estimates		$\frac{A}{C}$	$\frac{B}{C}$
		Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (A)	Simulation SE	Simulation Adj. SE (B)	Mean	Abs. % Bias	Finite Sample SE	Asymptotic SE (C)		
Threshold Parameters													
θ_1^1	-1.0000	-1.0110	1.10%	0.0600	0.0520	0.0023	0.0520	-1.0322	3.22%	0.0731	0.0545	0.9538	0.9548
θ_1^2	1.0000	0.9907	0.93%	0.0551	0.0515	0.0022	0.0515	1.0118	1.18%	0.0514	0.0528	0.9757	0.9766
θ_1^3	3.0000	3.0213	0.71%	0.0819	0.1177	0.0065	0.1179	2.9862	0.46%	0.1185	0.1188	0.9906	0.9921
θ_2^1	0.0000	-0.0234	-	0.0376	0.0435	0.0028	0.0436	0.0010	-	0.0418	0.0455	0.9572	0.9592
θ_2^2	2.0000	2.0089	0.44%	0.0859	0.0781	0.0066	0.0784	2.0371	1.86%	0.0949	0.0823	0.9491	0.9525
θ_3^1	-2.0000	-2.0266	1.33%	0.0838	0.0754	0.0060	0.0757	-2.0506	2.53%	0.0790	0.0776	0.9721	0.9752
θ_3^2	-0.5000	-0.5086	1.73%	0.0305	0.0440	0.0030	0.0441	-0.5090	1.80%	0.0378	0.0453	0.9702	0.9725
θ_3^3	1.0000	0.9917	0.83%	0.0516	0.0498	0.0035	0.0499	0.9987	0.13%	0.0569	0.0509	0.9774	0.9798
θ_3^4	2.5000	2.4890	0.44%	0.0750	0.0928	0.0066	0.0930	2.5148	0.59%	0.1144	0.0956	0.9699	0.9724
θ_4^1	1.0000	0.9976	0.24%	0.0574	0.0540	0.0050	0.0542	1.0255	2.55%	0.0656	0.0567	0.9526	0.9566
θ_4^2	3.0000	3.0101	0.34%	0.1107	0.1193	0.0125	0.1200	3.0048	0.16%	0.0960	0.1256	0.9498	0.9550
θ_5^1	-1.5000	-1.4875	0.84%	0.0694	0.0629	0.0056	0.0632	-1.5117	0.78%	0.0676	0.0649	0.9699	0.9737
θ_5^2	0.5000	0.4822	3.55%	0.0581	0.0465	0.0041	0.0467	0.4968	0.64%	0.0515	0.0472	0.9868	0.9906
θ_5^3	2.0000	1.9593	2.03%	0.0850	0.0741	0.0064	0.0744	2.0025	0.12%	0.0898	0.0761	0.9735	0.9771
Overall mean value across parameters		-	1.22%	0.0429	0.0428	0.0044	0.0432	-	1.28%	0.0455	0.0449	0.9493	0.9621

Simulation Results – The PMOP Model

Para-meter	True Value	MSL Approach: Parameter Estimates: Mean	MSL Approach: Parameter Estimates: Abs. % Bias	MSL Approach: Standard Error (SE) Estimates: Finite Sample SE	MSL Approach: Standard Error (SE) Estimates: Asymp-totic SE (A)	MSL Approach: Standard Error (SE) Estimates: Simula-tion SE	MSL Approach: Standard Error (SE) Estimates: Simula-tion Adj. SE (B)	CML Approach: Parameter Estimates: Mean	CML Approach: Parameter Estimates: Abs. % Bias	CML Approach: Standard Error (SE) Estimates: Finite Sample SE	CML Approach: Standard Error (SE) Estimates: Asymp-totic SE (C)	Rel. Eff.: $\frac{A}{C}$	Rel. Eff.: $\frac{B}{C}$
ρ = 0.30													
β_1	1.0000	0.9899	1.01%	0.1824	0.1956	0.0001	0.1956	0.9935	0.65%	0.1907	0.1898	1.0306	1.0306
β_2	1.0000	1.0093	0.93%	0.1729	0.1976	0.0001	0.1976	1.0221	2.21%	0.1955	0.2142	0.9223	0.9223
ρ	0.3000	0.2871	4.29%	0.0635	0.0605	0.0000	0.0605	0.2840	5.33%	0.0632	0.0673	0.8995	0.8995
σ^2	1.0000	1.0166	1.66%	0.2040	0.2072	0.0002	0.2072	1.0142	1.42%	0.2167	0.2041	1.0155	1.0155
θ^1	1.5000	1.5060	0.40%	0.2408	0.2615	0.0001	0.2615	1.5210	1.40%	0.2691	0.2676	0.9771	0.9771
θ^2	2.5000	2.5129	0.52%	0.2617	0.2725	0.0002	0.2725	2.5272	1.09%	0.2890	0.2804	0.9719	0.9719
θ^3	3.0000	3.0077	0.26%	0.2670	0.2814	0.0002	0.2814	3.0232	0.77%	0.2928	0.2882	0.9763	0.9763
Overall mean value across parameters		-	1.29%	0.1989	0.2109	0.0001	0.2109	-	1.84%	0.2167	0.2159	0.9705	0.9705
ρ = 0.70													
β_1	1.0000	1.0045	0.45%	0.2338	0.2267	0.0001	0.2267	1.0041	0.41%	0.2450	0.2368	0.9572	0.9572
β_2	1.0000	1.0183	1.83%	0.1726	0.1812	0.0001	0.1812	1.0304	3.04%	0.1969	0.2199	0.8239	0.8239
ρ	0.7000	0.6854	2.08%	0.0729	0.0673	0.0001	0.0673	0.6848	2.18%	0.0744	0.0735	0.9159	0.9159
σ^2	1.0000	1.0614	6.14%	0.4634	0.4221	0.0004	0.4221	1.0571	5.71%	0.4864	0.4578	0.9220	0.9220
θ^1	1.5000	1.5192	1.28%	0.2815	0.2749	0.0002	0.2749	1.5304	2.03%	0.3101	0.3065	0.8968	0.8968
θ^2	2.5000	2.5325	1.30%	0.3618	0.3432	0.0003	0.3432	2.5433	1.73%	0.3904	0.3781	0.9076	0.9076
θ^3	3.0000	3.0392	1.31%	0.4033	0.3838	0.0003	0.3838	3.0514	1.71%	0.4324	0.4207	0.9123	0.9123
Overall mean value across parameters		-	2.06%	0.2842	0.2713	0.0002	0.2713	-	2.40%	0.3051	0.2990	0.9051	0.9051

Simulation Results

- Non-convergence rates
  - The CMOP Model
    - Low correlation case: 28.5%
    - High correlation case: 35.5%
  - The PMOP Model
    - Low correlation case: 4.2%
    - High correlation case: 2.4%

Simulation Results

- Relative Computational Time Factor (RCTF)

  - The CMOP Model
    - Low correlation case: 18
    - High correlation case: 40

  - The PMOP Model
    - Low correlation case: 332
    - High correlation case: 231

Summary and Conclusions

- Compared the performance of the MSL approach with the CML approach in multivariate ordered-response situations
  - Cross-sectional setting, and
  - Panel setting
- Simulation data sets with known parameter vectors were used
- The results indicate that the CML approach recovers parameters as well as the MSL estimation approach
  - In addition, the ability of the CML approach to recover the parameters seems to be independent of the correlation structure

Summary and Conclusions

- The CML approach recovers parameters at a substantially reduced computational cost and improved computational stability
- Any reduction in the efficiency of the CML approach relative to the MSL approach is in the range of non-existent to small

Unordered Response Context

- "Workhorse" multinomial logit is saddled with the problem of IIA
- Several ways to relax the IID assumption
  - Multinomial Probit
  - GEV class of models
  - Mixed MNL
- Mixed MNL models are conceptually appealing
- These methods employ simulation based approaches to tackle integration within the likelihood function.
  - Accuracy of simulation techniques degrades rapidly at medium-to-high dimensions, and simulation noise increases convergence problems
  - Impractical in terms of computation time, or even infeasible, as the number of alternatives grows in the multinomial

Problem at Hand

- Consider a random utility formulation in which the utility that an individual *q* associates with alternative *i* (*i* = 1, 2, …, *I*) is written as:

$$U_{qi} = \beta' x_{qi} + \varepsilon_{qi}$$

- The probability of choosing alternative *m*

$$P_{qm} = \mathrm{Prob}[U_{qm} > U_{qi} \forall i \neq m] = \mathrm{Prob}[\beta' x_{qm} + \varepsilon_{qm} > \beta' x_{qi} + \varepsilon_{qi} \forall i \neq m]$$

- Alternatively $P_{qm} = \mathrm{Prob}[y^*_{qim} < 0 \forall i \neq m]$, where

$$y^*_{qim} = U_{qi} - U_{qm} = \beta' z_{qim} + \eta_{qim},\ z_{qim} = (x_{qi} - x_{qm}) \text{ and } \eta_{qim} = (\varepsilon_{qi} - \varepsilon_{qm})$$

Simulation Exercise – Cross-sectional MNP

Number of Runs : 50

Parameter	MNP MSL (150 Halton draws)					MNP MOPA				
	True Value	Mean Estimate	Mean Standard Error	Absolute Bias	Absolute Percentage Bias	True Value	Mean Estimate	Mean Standard Error	Absolute Bias	Absolute Percentage Bias
β_1	1.5000	1.3492	0.1254	0.1508	10.05	1.5000	1.5246	0.1855	0.0246	1.64
β_2	-1.0000	-0.8924	0.0860	0.1076	10.76	-1.0000	-1.0075	0.1250	0.0075	0.75
β_3	2.0000	1.7869	0.1635	0.2131	10.66	2.0000	2.0193	0.2434	0.0193	0.97
β_4	1.0000	0.8977	0.0866	0.1023	10.23	1.0000	1.0155	0.1262	0.0155	1.55
β_5	2.0000	-1.7977	0.1647	0.2023	10.12	2.0000	-2.0310	0.2443	0.0310	1.55
δ_1	1.0000	0.8929	0.1105	0.1071	10.71	1.0000	1.0147	0.1484	0.0147	1.47
δ_2	1.0000	0.8899	0.1079	0.1101	11.01	1.0000	1.0213	0.1495	0.0213	2.13
δ_3	1.0000	0.8756	0.1102	0.1244	12.44	1.0000	1.0012	0.1509	0.0012	0.12
δ_4	1.0000	0.8816	0.1091	0.1184	11.84	1.0000	1.0051	0.1477	0.0051	0.51
δ_5	1.0000	0.8952	0.1142	0.1048	10.48	1.0000	1.0173	0.1519	0.0173	1.73

Simulation Exercise – Cross-sectional MNP

Number of Runs : 50

Parameter	MNP MSL (150 Scrambled Halton draws)					MNP MOPA				
	True Value	Mean Estimate	Mean Standard Error	Absolute Bias	Absolute Percentage Bias	True Value	Mean Estimate	Mean Standard Error	Absolute Bias	Absolute Percentage Bias
β_1	1.5000	1.3395	0.1267	0.1605	10.70	1.5000	1.5246	0.1855	0.0246	1.64
β_2	-1.0000	-0.8866	0.0867	0.1134	11.34	-1.0000	-1.0075	0.1250	0.0075	0.75
β_3	2.0000	1.7731	0.1654	0.2269	11.35	2.0000	2.0193	0.2434	0.0193	0.97
β_4	1.0000	0.8900	0.0869	0.1100	11.00	1.0000	1.0155	0.1262	0.0155	1.55
β_5	2.0000	-1.7830	0.1662	0.2170	10.85	2.0000	-2.0310	0.2443	0.0310	1.55
δ_1	1.0000	0.8837	0.1077	0.1163	11.63	1.0000	1.0147	0.1484	0.0147	1.47
δ_2	1.0000	0.8814	0.1069	0.1186	11.86	1.0000	1.0213	0.1495	0.0213	2.13
δ_3	1.0000	0.8729	0.1103	0.1271	12.71	1.0000	1.0012	0.1509	0.0012	0.12
δ_4	1.0000	0.8680	0.1061	0.1320	13.20	1.0000	1.0051	0.1477	0.0051	0.51
δ_5	1.0000	0.8927	0.1114	0.1073	10.73	1.0000	1.0173	0.1519	0.0173	1.73

All these factors, combined with the conceptual and implementation simplicity of our approach, makes the approach promising

Thank You

Web Site:

http://www.ce.utexas.edu/prof/bhat